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Asymptotic behavior of solutions to a hyperbolic–elliptic coupled system in multi-dimensional radiating gas

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ABSTRACT

This paper is concerned with the asymptotic behavior of solutions to the Cauchy problem of a hyperbolic–elliptic coupled system in the multi-dimensional radiating gas

$$u_t + a \cdot \nabla u^2 + \operatorname{div} q = 0, \qquad -\nabla \operatorname{div} q + q + \nabla u = 0,$$

with initial data

$$u(x_1, \ldots, x_n, 0) = u_0(x_1, \ldots, x_n) \to u_\pm, \quad x_1 \to \pm\infty.$$

First, for the case with the same end states $u_- = u_+ = 0$, we prove the existence and uniqueness of the global solutions to the above Cauchy problem by combining some *a priori* estimates and the local existence based on the continuity argument. Then L^p-convergence rates of solutions are respectively obtained by applying L^2-energy method for $n = 1, 2, 3$ and L^p-energy method for $3 < n < 8$ and interpolation inequality. Furthermore, by semigroup argument, we obtain the decay rates to the diffusion waves for $1 \leqslant n < 8$. Secondly, for the case with the different end states $u_- < u_+$, our main concern is that the corresponding Cauchy problem in n-dimensional space $(n = 1, 2, 3)$ behaviors like planar rarefaction waves. Its convergence rate is also obtained by L^2-energy method and L^1-estimate.

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1. Introduction and main results

In this paper, we are concerned with the asymptotic behavior of solutions to the Cauchy problem of the hyperbolic–elliptic coupled system in the multi-dimensional radiating gas

$$\begin{cases} u_t + a \cdot \nabla u^2 + \operatorname{div} q = 0, \\ -\nabla \operatorname{div} q + q + \nabla u = 0, \end{cases} \tag{1.1}$$

with initial data

$$u(x,0) = u(x_1, \ldots, x_n, 0) = u_0(x_1, \ldots, x_n) \to u_\pm, \quad x_1 \to \pm\infty, \tag{1.2}$$

where $u_\pm$ are given constant states, $a \in \mathbb{R}^n$ is a constant vector and u, q are unknown functions of the spacial variable $x \in \mathbb{R}^n$ and the time variable t. Typically, $u = u(x,t)$ and $q = (q_1, \ldots, q_n)(x,t)$ represent the velocity and radiating heat flux of the gas respectively.

The system (1.1) is a simplified version of the model for the motion of radiating gas in n-dimensional space. More precisely, in a certain physical situation, the system (1.1) gives a good approximation to the fundamental system describing the motion of a radiating gas, which is a quite general model for compressible gas dynamics where heat radiative transfer phenomena are taken into account and given by the hyperbolic–elliptic coupled model

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u + pI) = 0, \\ \left\{\rho\left(e + \dfrac{|u|^2}{2}\right)\right\}_t + \operatorname{div}\left\{\rho u\left(e + \dfrac{|u|^2}{2}\right) + pu + q\right\} = 0, \\ -\nabla \operatorname{div} q + \lambda q + \mu \nabla \theta^4 = 0, \end{cases} \tag{1.3}$$

where ρ, u, p, e and θ are respectively the mass density, velocity, pressure, internal energy and absolute temperature of the gas, while q is the radiative heat flux, and λ and μ are given positive constants depending on the gas itself. The first three equations are motivated as for the usual Euler system. The study of the Euler equations is a classical topic. However the physical motivation of the fourth equation, which take into account of heat radiation phenomena, is given in [24]. On the other hand, The simplified model (1.1) was first recovered by Hamer (see [6]), and for the reduction of system (1.3) to system (1.1), we refer to [4,6,24].

The system (1.1) has been extensively studied by several authors in different contexts recently, but most of which are in the case of one space dimension.

Concerning the large-time behavior of solutions to the Cauchy problem

$$\begin{cases} u_t + uu_{x_1} + q_{x_1} = 0, \\ -q_{x_1x_1} + q + u_{x_1} = 0, \\ u(x_1, 0) = u_0(x_1) \to u_\pm, \quad \text{as } x_1 \to \pm\infty, \end{cases} \tag{1.4}$$

we mention [23,13,14]. In [23], Tanaka discussed the case where $u_- = u_+ = 0$, and proved that the solution to the Cauchy problem (1.4) approaches the diffusion wave which is the self-similar solution to the viscous Burgers equation $u_t + uu_{x_1} = u_{x_1x_1}$. In [13], Kawashima and Nishibata discussed the case where $u_- > u_+$, and proved that the solution to the Cauchy problem (1.4) approaches the traveling wave of shock profile. The remaining case $u_- < u_+$ was studied in [14]. In this case, Kawashima and Tanaka showed the asymptotic stability of the rarefaction waves for the Cauchy problem (1.4) and obtained the convergence rates.

In addition, let $\psi(x)$ be the fundamental solution to the elliptic operator $-\triangle + I$ in $\mathbb{R}^n$ $(n \geqslant 1)$. Then one can solve the second equation $(1.1)_2$ to obtain $\operatorname{div} q$ by u and substitute it into the first equation $(1.1)_1$ to rewrite (1.1) as a scalar balance law of the form

$$u_t + a \cdot \nabla u^2 = -u + \psi * u, \tag{1.5}$$

which is the most convenient approach to obtain the local existence, L^1-estimate and L^p-estimate $(p > 2)$ for the solutions to the Cauchy problem (1.1), (1.2). Here "$*$" denotes the convolution with respect to the space variable x. As it is well-known [6,15,22], the scalar equation is equivalent to the system (1.1). In one space dimension ($a = \frac{1}{2}$), Eq. (1.5) was also studied in [21,16,15,26,3].

In the multi-dimensional case, Francesco [2] obtained the global well-posedness of the system (1.1) and analyzed the relaxation limits. Recently, for the Cauchy problem of a model system of the radiating gas in two space dimensions, Gao and Zhu [4] investigated the asymptotic decay rates toward the planar rarefaction waves based on L^2-energy method. More recently, based on L^p-energy method, Gao, Ruan and Zhu [5] studied decay rates to the planar rarefaction waves for the n-dimensional model system of the radiating gas for $n = 3, 4, 5$. In addition, there are a lot of related works concerning the stability of rarefaction waves, viscous shock waves and diffusion waves for viscous conservation laws and other system, we refer to [9,10,17,18,20,25,27,29] and references therein.

In this paper, for simplicity, without loss of generality, we choose $a = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^n$ in (1.1). That is, we will consider the following Cauchy problem

$$\begin{cases} u_t + \displaystyle\sum_{i=1}^{n} u u_{x_i} + \operatorname{div} q = 0, \\ -\nabla \operatorname{div} q + q + \nabla u = 0, \end{cases} \tag{1.6}$$

with initial data

$$u(x_1, \dots, x_n, 0) = u_0(x_1, \dots, x_n) \to u_\pm, \quad x_1 \to \pm\infty. \tag{1.7}$$

Besides the equivalent system (1.5), we fortunately find that the system (1.6) can be transformed into the following high-order scalar equation of conservation law from $(1.6)_1 - (\triangle(1.6)_1 + \operatorname{div}(1.6)_2)$

$$u_t + \sum_{i=1}^{n} u u_{x_i} - \triangle u_t - \triangle\left(\sum_{i=1}^{n} u u_{x_i}\right) - \triangle u = 0, \tag{1.8}$$

which will make calculations more simple and direct when we obtain L^2-estimates, especially high-order L^2-estimates.

In what follows, we will claim that the system (1.6) is truly equivalent to Eq. (1.8). In fact, first, let $(u, q)(x, t)$ be a solution to (1.6). Then it is easy to verify $u(x, t)$ is a solution of (1.8) by $(1.6)_1 - (\triangle(1.6)_1 + \operatorname{div}(1.6)_2)$. On the other hand, let $u(x, t)$ be a solution of (1.8). Define

$$q = -\nabla u_t - \nabla\left(\sum_{i=1}^{n} u u_{x_i}\right) - \nabla u.$$

Then

$$\operatorname{div} q = -\triangle u_t - \triangle\left(\sum_{i=1}^{n} u u_{x_i}\right) - \triangle u.$$

This and (1.8) imply $(1.6)_1$. Furthermore

$$-\nabla \operatorname{div} q + q + \nabla u = -\nabla\left\{u_t + \sum_{i=1}^{n} u u_{x_i} - \triangle u_t - \triangle\left(\sum_{i=1}^{n} u u_{x_i}\right) - \triangle u\right\} = 0.$$

Thus $(u, q)(x, t)$ is a solution to (1.6). These show that the system (1.6) is truly equivalent to Eq. (1.8).

Notations. Hereafter, we denote all positive constants by only C without confusion. For function spaces, $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ denote usual Lebesgue spaces in $\mathbb{R}^n$ respectively with its norm $\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p\,dx)^{1/p}$, $1 \leqslant p < \infty$, and $\|f\|_{L^\infty(\mathbb{R}^n)} = \sup_{\mathbb{R}^n} |f(x)|$. We also write $\|\cdot\|_{L^p(\mathbb{R}^n)} = |\cdot|_p$, $1 \leqslant p \leqslant \infty$. $H^k(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$ denote the usual k-th order Sobolev spaces with its norm $\|f\|_k = \|f\|_{H^k} = (\sum_{i=0}^k |\nabla^i f|_2^2)^{1/2}$ and $\|f\|_{W^{k,p}} = (\sum_{i=0}^k |\nabla^i f|_p^p)^{1/p}$ respectively. For simplicity, $|f(\cdot,t)|_p$, $\|f(\cdot,t)\|_k$ and $\|f(\cdot,t)\|_{W^{k,p}}$ are denoted by $|f(t)|_p$, $\|f(t)\|_k$ and $\|f(t)\|_{W^{k,p}}$ respectively. When $f = f(x)$ and $k = k_1 + \cdots + k_n$, we also use symbols: $\nabla^k f = (\partial_{x_1}^k f, \partial_{x_1}^{(k-1)}\partial_{x_2} f, \ldots, \partial_{x_1}^{k_1}\cdots\partial_{x_n}^{k_n} f, \ldots, \partial_{x_{n-1}}\partial_{x_n}^{(k-1)} f, \partial_{x_n}^k f)$, $\nabla^0 f = f$.

Now, we will state our main results. First, for the case with the same end states $u_- = u_+ = 0$, we have the following results for the Cauchy problem (1.6), (1.7).

Theorem 1.1. *Let $u_- = u_+ = 0$, $u_0 \in H^3(\mathbb{R}^n)$ $(n = 1, 2, 3)$ and $\|u_0\|_3$ be sufficiently small. Then the Cauchy problem* (1.6), (1.7) *admits a unique global solution $(u, q)(x,t)$ satisfying*

$$\begin{gathered} u \in C^0\big([0,\infty); H^3\big(\mathbb{R}^n\big)\big), \qquad \nabla u \in L^2\big([0,\infty); H^2\big(\mathbb{R}^n\big)\big), \\ q \in C^0\big([0,\infty); H^4\big(\mathbb{R}^n\big)\big) \cap L^2\big([0,\infty); H^4\big(\mathbb{R}^n\big)\big). \end{gathered}$$

Theorem 1.2. *Under the same assumptions of Theorem* 1.1, *furthermore, if $u_0 \in L^1(\mathbb{R}^n)$, then the global solution $u(x,t)$ obtained in Theorem* 1.1 *satisfies the following decay estimates for $t \geqslant 0$:*

$$\begin{cases} \big|u(t)\big|_p \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, \quad 1 \leqslant p \leqslant \infty, \\ \big|\nabla u(t)\big|_p \leqslant \begin{cases} C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, & 2 \leqslant p \leqslant \infty,\ n=1, \\ C(1+t)^{-\frac{(3n+4)p-2n}{8p}}, & 2 \leqslant p \leqslant \infty,\ n = 2,3. \end{cases} \end{cases} \tag{1.9}$$

Remark 1.1. In (1.9), when $n = 1$, the decay rate on $\|u(t)\|_{W^{1,p}}$ $(2 \leqslant p \leqslant \infty)$ is optimal.

Theorem 1.3. *Let $u_- = u_+ = 0$, $u_0 \in (L^1 \cap H^3 \cap W^{3,4})(\mathbb{R}^n)$ $(3 < n < 8)$ and $|u_0|_1 + \|u_0\|_3 + \|u_0\|_{W^{3,4}}$ be sufficiently small. Then the Cauchy problem* (1.6), (1.7) *admits a unique global solution $(u,q)(x,t)$ satisfying*

$$\begin{gathered} u \in C^0\big([0,\infty); \big(H^3 \cap W^{3,4}\big)\big(\mathbb{R}^n\big)\big), \qquad \nabla u \in L^2\big([0,\infty); \big(H^2 \cap W^{2,4}\big)\big(\mathbb{R}^n\big)\big), \\ q \in C^0\big([0,\infty); \big(H^4 \cap W^{3,4}\big)\big(\mathbb{R}^n\big)\big) \cap L^2\big([0,\infty); \big(H^4 \cap W^{3,4}\big)\big(\mathbb{R}^n\big)\big). \end{gathered}$$

In addition, the solution $u(x,t)$ verifies the following decay estimates for $t \geqslant 0$:

$$\begin{cases} \big|u(t)\big|_p \leqslant \begin{cases} C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 1 \leqslant p \leqslant 2, \\ C(1+t)^{-\frac{n(3p-2)}{8p}}, & 2 \leqslant p \leqslant \infty, \end{cases} \\ \big|\nabla u(t)\big|_p \leqslant C(1+t)^{-\frac{(5n+8)p-2n}{16p}}, \quad 2 \leqslant p \leqslant \infty. \end{cases} \tag{1.10}$$

Theorem 1.4. *Under the same conditions of Theorem* 1.2 (*or Theorem* 1.3), *there exists an asymptotic profile $G(x,t)$* (*defined by* (2.3.9)) *of u, such that for $t \geqslant 0$*

$$\big|(u-G)(t)\big|_2 \leqslant \begin{cases} C(1+t)^{-\frac{1}{4}}, & n = 1, \\ C(1+t)^{-1}\ln(1+t), & n = 2, \\ C(1+t)^{-(\frac{n}{4}+\frac{1}{2})}, & 3 \leqslant n < 8. \end{cases} \tag{1.11}$$

Secondly, we state the main theorem for the case with the different end states $u_- < u_+$.

Theorem 1.5. *Let $u_- < u_+$, $z_0 - u_0^R \in (L^1 \cap H^4)(\mathbb{R})$, $u_0 - z_0 \in (L^1 \cap H^3)(\mathbb{R}^n)$ $(n = 2, 3)$, $z_0'(x_1) > 0$ and $|z_0 - u_0^R|_1 + \|z_0 - u_0^R\|_4 + \|u_0 - z_0\|_3 + \delta_0$ be sufficiently small. Then the Cauchy problem* (1.6), (1.7) *admits a unique global solution $u(x,t)$ satisfying the following decay estimate for sufficiently large $t > 0$:*

$$\left|u(\cdot,t) - u^R(\cdot/t)\right|_\infty \leqslant Ct^{-\frac{1}{2}}. \tag{1.12}$$

Here $\delta_0 = u_+ - u_- > 0$, u_0^R, $u^R(\cdot/t) = u^R(x_1/t)$ and z_0 are defined by (3.0.2) *and* (3.1.1) *later, respectively.*

Remark 1.2. The decay rate (1.12) improves ones in [4] for $n = 2$ and [5] for $n = 3$.

Finally, we cite the following fundamental lemma which will frequently be used later.

Lemma 1.1. *(See [2,19].) The fundamental solution $\psi(x)$ of the elliptic operator $-\triangle + I$ in $\mathbb{R}^n$ $(n \geqslant 1)$ satisfies the following properties*

$$\begin{cases} \psi(x) \geqslant 0, \quad \displaystyle\int_{\mathbb{R}^n} \psi(x)\,dx = 1, \\ \psi(x) \in L^s\big(\mathbb{R}^n\big) \quad \text{for all } 1 \leqslant s < \dfrac{n}{n-2}\ (n \geqslant 3). \end{cases} \tag{1.13}$$

This paper is arranged as follows. In Section 2, when $1 \leqslant n < 8$, we will give the global existence and decay estimates toward the diffusion waves for the case with the same end states $u_- = u_+ = 0$. In Section 3, when $n = 1, 2, 3$, we will establish asymptotic stability toward the planar rarefaction waves for the case with the different end states $u_- < u_+$ and the convergence rates are also obtained.

2. The case with the same end states $u_- = u_+ = 0$

In this section, we consider the following equivalent Cauchy problem corresponding to (1.6), (1.7) with the same end states $u_- = u_+ = 0$. That is

$$\begin{cases} u_t + \displaystyle\sum_{i=1}^n u u_{x_i} - \triangle u_t - \triangle\left(\sum_{i=1}^n u u_{x_i}\right) - \triangle u = 0, \\ u(x,0) = u_0(x) \to 0, \quad x_1 \to \pm\infty. \end{cases} \tag{2.0.1}$$

2.1. The case of $n = 1, 2, 3$

In this subsection, for the case of $n = 1, 2, 3$, we seek the solutions of (2.0.1) in the solution space $X_1(0,T)$ defined by

$$X_1(0,T) = \left\{u \in C^0\big([0,T); H^3\big(\mathbb{R}^n\big)\big),\ \nabla u \in L^2\big([0,T); H^2\big(\mathbb{R}^n\big)\big)\right\} \tag{2.1.1}$$

for $0 < T \leqslant \infty$ under *a priori* assumptions

$$\left|u(t)\right|_\infty \leqslant C\varepsilon_1, \qquad \left|\nabla u(t)\right|_\infty \leqslant C\varepsilon_1, \quad 0 < \varepsilon_1 \ll 1. \tag{2.1.2}$$

First, the local existence of solutions to the Cauchy problem (2.0.1) is stated as follows:

Lemma 2.1. *Suppose that $u_0 \in H^3(\mathbb{R}^n)$. Then there is a positive constant T_0 depending on $\|u_0\|_3$ such that the Cauchy problem* (2.0.1) *admits a unique solution $u(x,t) \in X_1(0,T_0)$.*

This lemma can be proved by the following equivalent Cauchy problem

$$\begin{cases} u_t + \displaystyle\sum_{i=1}^n u u_{x_i} + u - \psi * u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1.3}$$

The proof is omitted since the procedure is standard as in [11].

Next, in order to obtain the global existence of the solution to the Cauchy problem (2.0.1), we devote ourselves to the *a priori* estimates given by two lemmas below.

Lemma 2.2. *Let the assumptions in Theorem* 1.1 *hold. Then the solution $u(x,t)$ of* (2.0.1) *satisfies for $k = 1, 2, 3$*

$$\left\|\nabla^{k-1} u(t)\right\|_1^2 + \int_0^t \left|\nabla^k u(\tau)\right|_2^2 d\tau \leqslant C\|u_0\|_k^2.$$

Proof. The proof is divided into three steps.

Step 1 (The case with $k=1$). Multiplying $(2.0.1)_1$ by $2u$, we have

$$\left(u^2 + |\nabla u|^2\right)_t + 2|\nabla u|^2 - 2u\triangle\left(u\sum_{i=1}^n u_{x_i}\right) - 2\operatorname{div}\left\{u\nabla(u+u_t)\right\} + \frac{2}{3}\sum_{i=1}^n \left(u^3\right)_{x_i} = 0. \tag{2.1.4}$$

Since $\nabla(fg) = g\nabla f + f\nabla g$, $\operatorname{div}(f\vec{g}) = f\operatorname{div}\vec{g} + \nabla f\cdot\vec{g}$ with $\vec{g} = (g_1, g_2, \ldots, g_n)$, then

$$\begin{aligned} -2u\triangle\left(u\sum_{i=1}^n u_{x_i}\right) &= -2\operatorname{div}\left\{u\nabla\left(u\sum_{i=1}^n u_{x_i}\right)\right\} + 2\nabla u\cdot\nabla\left(u\sum_{i=1}^n u_{x_i}\right) \\ &= -2\operatorname{div}\left\{u\nabla\left(u\sum_{i=1}^n u_{x_i}\right)\right\} + |\nabla u|^2\sum_{i=1}^n u_{x_i} + \sum_{i=1}^n\left(u|\nabla u|^2\right)_{x_i}. \end{aligned} \tag{2.1.5}$$

Substituting (2.1.5) into (2.1.4), then integrating the resulting equation over $\mathbb{R}^n$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left(u^2 + |\nabla u|^2\right)dx + 2\int_{\mathbb{R}^n}|\nabla u|^2\,dx = -\int_{\mathbb{R}^n}|\nabla u|^2\sum_{i=1}^n u_{x_i}\,dx. \tag{2.1.6}$$

From (2.1.2), we arrive at

$$I_1 = -\int_{\mathbb{R}^n}|\nabla u|^2\sum_{i=1}^n u_{x_i}\,dx \leqslant C\left|\nabla u(t)\right|_\infty\int_{\mathbb{R}^n}|\nabla u|^2\,dx \leqslant C\varepsilon_1\int_{\mathbb{R}^n}|\nabla u|^2\,dx. \tag{2.1.7}$$

Substituting (2.1.7) into (2.1.6) and integrating the resulting equation over $(0,t)$, we obtain

$$\int_{\mathbb{R}^n}\left(u^2+|\nabla u|^2\right)dx+\int_0^t\int_{\mathbb{R}^n}|\nabla u|^2\,dx\,d\tau\leqslant C\|u_0\|_1^2. \tag{2.1.8}$$

Step 2 (The case with $k=2$). Multiplying $(2.0.1)_1$ by $-2\triangle u$, and using the calculations similar to (2.1.5), we have

$$\begin{aligned}&\left(|\nabla u|^2+|\triangle u|^2\right)_t+2|\triangle u|^2+|\nabla u|^2\sum_{i=1}^n u_{x_i}+2\triangle u\triangle\left(u\sum_{i=1}^n u_{x_i}\right)\\&\quad-2\operatorname{div}\left\{\left(u_t+\sum_{i=1}^n uu_{x_i}\right)\nabla u\right\}+\sum_{i=1}^n\left\{u|\nabla u|^2\right\}_{x_i}=0.\end{aligned} \tag{2.1.9}$$

Since $\triangle(fg)=f\triangle g+2\nabla f\cdot\nabla g+g\triangle f$, then

$$\begin{aligned}2\triangle u\triangle\left(u\sum_{i=1}^n u_{x_i}\right)&=2\triangle u\sum_{i=1}^n(u_{x_i}\triangle u+2\nabla u\cdot\nabla u_{x_i}+u\triangle u_{x_i})\\&=|\triangle u|^2\sum_{i=1}^n u_{x_i}+4\triangle u\sum_{i=1}^n\nabla u\cdot\nabla u_{x_i}+\sum_{i=1}^n\left\{u|\triangle u|^2\right\}_{x_i}.\end{aligned} \tag{2.1.10}$$

Substituting (2.1.10) into (2.1.9), then integrating the resulting equation over $\mathbb{R}^n$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\nabla u|^2+|\triangle u|^2\right)dx+2\int_{\mathbb{R}^n}|\triangle u|^2\,dx\\&\quad=I_1-\int_{\mathbb{R}^n}|\triangle u|^2\sum_{i=1}^n u_{x_i}\,dx-4\int_{\mathbb{R}^n}\triangle u\sum_{i=1}^n\nabla u\cdot\nabla u_{x_i}\,dx=\sum_{i=1}^3 I_i.\end{aligned} \tag{2.1.11}$$

From (2.1.2), we arrive at

$$I_2\leqslant C\left|\nabla u(t)\right|_\infty\int_{\mathbb{R}^n}|\triangle u|^2\,dx\leqslant C\varepsilon_1\int_{\mathbb{R}^n}|\triangle u|^2\,dx. \tag{2.1.12}$$

Direct calculations show

$$\begin{aligned}|\triangle u|^2&=\left(\sum_{i=1}^n u_{x_ix_i}\right)^2=\sum_{i=1}^n u_{x_ix_i}^2+2\sum_{1\leqslant i<j\leqslant n}u_{x_ix_i}u_{x_jx_j}\\&=\sum_{i=1}^n u_{x_ix_i}^2+2\sum_{1\leqslant i<j\leqslant n}u_{x_ix_j}^2+2\sum_{1\leqslant i<j\leqslant n}\left[(u_{x_i}u_{x_jx_j})_{x_i}-(u_{x_i}u_{x_ix_j})_{x_j}\right]\\&=\sum_{i,j=1}^n u_{x_ix_j}^2+2\sum_{1\leqslant i<j\leqslant n}\left[(u_{x_i}u_{x_jx_j})_{x_i}-(u_{x_i}u_{x_ix_j})_{x_j}\right],\end{aligned} \tag{2.1.13}$$

which means

$$\left|\triangle u(t)\right|_2^2=\int_{\mathbb{R}^n}|\triangle u|^2\,dx=\sum_{i=1}^n\sum_{j=1}^n\int_{\mathbb{R}^n}u_{x_ix_j}^2\,dx\geqslant\int_{\mathbb{R}^n}\left|\nabla^2 u\right|^2dx=\left|\nabla^2 u(t)\right|_2^2. \tag{2.1.14}$$

(2.1.2) and (2.1.14) imply

$$I_3\leqslant 2\left|\nabla u(t)\right|_\infty\sum_{i=1}^n\left(\left|\nabla u_{x_i}(t)\right|_2^2+\left|\triangle u(t)\right|_2^2\right)\leqslant C\varepsilon_1\left|\triangle u(t)\right|_2^2. \tag{2.1.15}$$

Substituting (2.1.12) and (2.1.15) into (2.1.11), then integrating the resulting equation over $(0,t)$, we get by (2.1.7) and (2.1.8)

$$\int_{\mathbb{R}^n}\left(|\nabla u|^2+|\triangle u|^2\right)dx+\int_0^t\int_{\mathbb{R}^n}|\triangle u|^2\,dx\,d\tau\leqslant C\|u_0\|_2^2. \tag{2.1.16}$$

(2.1.14) and (2.1.16) yield

$$\int_{\mathbb{R}^n}\left(|\nabla u|^2+\left|\nabla^2 u\right|^2\right)dx+\int_0^t\int_{\mathbb{R}^n}\left|\nabla^2 u\right|^2dx\,d\tau\leqslant C\|u_0\|_2^2. \tag{2.1.17}$$

Step 3 (The case with $k=3$). Multiplying $\triangle(2.0.1)_1$ by $2\triangle u$ and using (2.1.10), we have

$$\begin{aligned}&\left(|\triangle u|^2+\left|\nabla(\triangle u)\right|^2\right)_t+2\left|\nabla(\triangle u)\right|^2+|\triangle u|^2\sum_{i=1}^n u_{x_i}+4\triangle u\sum_{i=1}^n\nabla u\cdot\nabla u_{x_i}\\&\quad-2\triangle u\triangle^2\left(u\sum_{i=1}^n u_{x_i}\right)-2\,\mathrm{div}\left\{\triangle u\nabla(\triangle u+\triangle u_t)\right\}+\sum_{i=1}^n\left\{u|\triangle u|^2\right\}_{x_i}=0.\end{aligned} \tag{2.1.18}$$

Rewrite the fifth term on the left-hand side of (2.1.18) as follows:

$$\begin{aligned}-2\triangle u\triangle^2\left(u\sum_{i=1}^n u_{x_i}\right)&=-2\triangle u\triangle\left\{\sum_{i=1}^n\left(u_{x_i}\triangle u+2\nabla u\cdot\nabla u_{x_i}+u\triangle u_{x_i}\right)\right\}\\&=K_1+K_2+K_3.\end{aligned} \tag{2.1.19}$$

Further rewrite K_1,K_2 and K_3 as follows:

$$\begin{aligned}K_1&=-2\,\mathrm{div}\left\{\triangle u\nabla\left(\triangle u\sum_{i=1}^n u_{x_i}\right)\right\}+2\nabla(\triangle u)\cdot\nabla\left(\triangle u\sum_{i=1}^n u_{x_i}\right)\\&=-2\,\mathrm{div}\left\{\triangle u\nabla\left(\triangle u\sum_{i=1}^n u_{x_i}\right)\right\}+2\left|\nabla(\triangle u)\right|^2\sum_{i=1}^n u_{x_i}+2\sum_{i,j=1}^n u_{x_ix_j}\triangle u\triangle u_{x_i},\end{aligned} \tag{2.1.20}$$

$$\begin{aligned} K_2 &= -2\operatorname{div}\left\{\triangle u\nabla\left(\sum_{i=1}^n\left(|\nabla u|^2\right)_{x_i}\right)\right\} + 2\nabla(\triangle u)\cdot\nabla\left(\sum_{i=1}^n\left(|\nabla u|^2\right)_{x_i}\right) \\ &= -2\operatorname{div}\left\{\triangle u\nabla\left(\sum_{i=1}^n\left(|\nabla u|^2\right)_{x_i}\right)\right\} + 4\sum_{i,j,k=1}^n (u_{x_ix_k}u_{x_jx_k} + u_{x_k}u_{x_ix_jx_k})\triangle u_{x_j}, \end{aligned} \tag{2.1.21}$$

$$\begin{aligned} K_3 &= -2\operatorname{div}\left\{\triangle u\nabla\left(u\sum_{i=1}^n\triangle u_{x_i}\right)\right\} + 2\nabla(\triangle u)\cdot\nabla\left(u\sum_{i=1}^n\triangle u_{x_i}\right) \\ &= -2\operatorname{div}\left\{\triangle u\nabla\left(u\sum_{i=1}^n\triangle u_{x_i}\right)\right\} + \sum_{i=1}^n\left\{u\left|\nabla(\triangle u)\right|^2\right\}_{x_i} \\ &\quad - \left|\nabla(\triangle u)\right|^2\sum_{i=1}^n u_{x_i} + 2\sum_{i,j=1}^n u_{x_j}\triangle u_{x_i}\triangle u_{x_j}. \end{aligned} \tag{2.1.22}$$

Substituting (2.1.19)–(2.1.22) into (2.1.18), then integrating the resulting equation over $\mathbb{R}^n$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\triangle u|^2 + \left|\nabla(\triangle u)\right|^2\right)dx + 2\int_{\mathbb{R}^n}\left|\nabla(\triangle u)\right|^2 dx = \sum_{i=2}^6 I_i, \tag{2.1.23}$$

where I_2 and I_3 are defined by (2.1.11) and

$$I_4 = -\int_{\mathbb{R}^n}\left|\nabla(\triangle u)\right|^2\sum_{i=1}^n u_{x_i}\,dx, \qquad I_5 = -2\int_{\mathbb{R}^n}\sum_{i,j=1}^n (u_{x_ix_j}\triangle u + u_{x_j}\triangle u_{x_j})\triangle u_{x_i}\,dx,$$

$$I_6 = -4\int_{\mathbb{R}^n}\sum_{i,j,k=1}^n (u_{x_ix_k}u_{x_jx_k} + u_{x_k}u_{x_ix_jx_k})\triangle u_{x_j}\,dx.$$

From (2.1.2), we obtain

$$I_4 \leqslant C\left|\nabla u(t)\right|_\infty\int_{\mathbb{R}^n}\left|\nabla(\triangle u)\right|^2 dx \leqslant C\varepsilon_1\int_{\mathbb{R}^n}\left|\nabla(\triangle u)\right|^2 dx. \tag{2.1.24}$$

Next, we estimate I_5 and I_6, which are reduced to the terms $I_7 = \int_{\mathbb{R}^n} u_{x_{i_1}x_{i_2}}u_{x_{j_1}x_{j_2}}u_{x_{k_1}x_{k_2}x_{k_3}}\,dx$ and $I_8 = \int_{\mathbb{R}^n} u_{x_{i_1}}u_{x_{j_1}x_{j_2}x_{j_3}}u_{x_{k_1}x_{k_2}x_{k_3}}\,dx$, where $i_1, i_2, j_1, j_2, j_3, k_1, k_2, k_3 \in \{1, 2, 3\}$.

By the Cauchy and Gagliardo–Nirenberg inequalities, I_7 is bounded as follows:

$$I_7 \leqslant \frac{1}{8}\int_{\mathbb{R}^n}\left|\nabla^3 u\right|^2 dx + C\int_{\mathbb{R}^n}\left|\nabla^2 u\right|^4 dx \leqslant \left(\frac{1}{8} + C\varepsilon_1\right)\int_{\mathbb{R}^n}\left|\nabla^3 u\right|^2 dx. \tag{2.1.25}$$

In addition, it is easy to get

$$I_8 \leqslant C\left|\nabla u(t)\right|_\infty\int_{\mathbb{R}^n}\left(u^2_{x_{j_1}x_{j_2}x_{j_3}} + u^2_{x_{k_1}x_{k_2}x_{k_3}}\right)dx \leqslant C\varepsilon_1\int_{\mathbb{R}^n}\left|\nabla^3 u\right|^2 dx. \tag{2.1.26}$$

Substituting (2.1.24)–(2.1.26) into (2.1.23), then integrating the resulting equation over $(0,t)$, we obtain by (2.1.12), (2.1.15) and (2.1.16)

$$\int_{\mathbb{R}^n} \left(|\triangle u|^2 + \left|\nabla(\triangle u)\right|^2\right) dx + \int_0^t \int_{\mathbb{R}^n} \left|\nabla(\triangle u)\right|^2 dx\, d\tau \leqslant \left(\frac{1}{2} + C\varepsilon_1\right) \int_{\mathbb{R}^n} \left|\nabla^3 u\right|^2 dx + C\|u_0\|_3^2. \tag{2.1.27}$$

Direct calculations show

$$\left|\nabla(\triangle u)\right|^2 = \sum_{i,j,k=1}^{n} u_{x_i x_j x_k}^2 + 2\sum_{k=1}^{n} \sum_{1\leqslant i<j\leqslant n} \left[(u_{x_i x_k} u_{x_j x_j x_k})_{x_i} - (u_{x_i x_k} u_{x_i x_j x_k})_{x_j}\right],$$

which means

$$\int_{\mathbb{R}^n} \left|\nabla(\triangle u)\right|^2 dx = \sum_{i,j,k=1}^{n} \int_{\mathbb{R}^n} u_{x_i x_j x_k}^2\, dx \geqslant \int_{\mathbb{R}^n} \left|\nabla^3 u\right|^2 dx. \tag{2.1.28}$$

(2.1.27) and (2.1.28) imply

$$\int_{\mathbb{R}^n} \left(|\nabla u|^2 + \left|\nabla^3 u\right|^2\right) dx + \int_0^t \int_{\mathbb{R}^n} \left|\nabla^3 u\right|^2 dx\, d\tau \leqslant C\|u_0\|_3^2. \tag{2.1.29}$$

This completes the proof of Lemma 2.2. □

With the help of Lemma 2.2, we can give the *a priori* estimates on $q(x,t)$.

Lemma 2.3. *Let the assumptions in Theorem* 1.1 *hold. Then the solution* $q(x,t)$ *of* (1.6), (1.7) *satisfies for* $l = 1, 2, 3$

$$\left\|q(t)\right\|_{l+1} \leqslant C\|u_0\|_l. \tag{2.1.30}$$

Proof. The proof is divided into four steps.

Step 1. Integrating the resulting equation from $q \cdot (1.6)_2$ over $\mathbb{R}^n$, we have

$$-\int_{\mathbb{R}^n} q\cdot\nabla \operatorname{div} q\, dx + \int_{\mathbb{R}^n} |q|^2\, dx - \int_{\mathbb{R}^n} q\cdot\nabla u\, dx \leqslant \frac{1}{2}\int_{\mathbb{R}^n} |q|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

By using the divergence theorem, we get

$$2\int_{\mathbb{R}^n} |\operatorname{div} q|^2\, dx + \int_{\mathbb{R}^n} |q|^2\, dx \leqslant \int_{\mathbb{R}^n} |\nabla u|^2\, dx,$$

which together with (2.1.8) implies $\|q(t)\|_1 \leqslant C\|u_0\|_1$.

Step 2. Integrating the resulting equation from $-\nabla \operatorname{div} q \cdot (1.6)_2$ over $\mathbb{R}^n$, by using divergence theorem and the Cauchy inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla \operatorname{div} q|^2\,dx + \int_{\mathbb{R}^n} |\operatorname{div} q|^2\,dx &= \int_{\mathbb{R}^n} \nabla \operatorname{div} q \cdot \nabla u\,dx\\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \operatorname{div} q|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx,\end{aligned}$$

which implies by (2.1.8) $\|q(t)\|_2 \leqslant C\|u_0\|_1$.

Step 3. Integrating the resulting equation from $-\triangle \operatorname{div} q \times \operatorname{div}(1.6)_2$ over $\mathbb{R}^n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\triangle \operatorname{div} q|^2\,dx &= \int_{\mathbb{R}^n} (\triangle \operatorname{div} q) \operatorname{div} q\,dx + \int_{\mathbb{R}^n} (\triangle \operatorname{div} q)\triangle u\,dx\\ &= \int_{\mathbb{R}^n} \left\{\operatorname{div}(\operatorname{div} q \nabla \operatorname{div} q) - |\nabla \operatorname{div} q|^2\right\} dx + \int_{\mathbb{R}^n} (\triangle \operatorname{div} q)\triangle u\,dx.\end{aligned}$$

By the Cauchy inequality, we get

$$\int_{\mathbb{R}^n} |\triangle \operatorname{div} q|^2\,dx + \int_{\mathbb{R}^n} |\nabla \operatorname{div} q|^2\,dx \leqslant \frac{1}{2}\int_{\mathbb{R}^n} |\triangle \operatorname{div} q|^2\,dx + \int_{\mathbb{R}^n} |\triangle u|^2\,dx,$$

which implies by (2.1.16) $\|q(t)\|_3 \leqslant C\|u_0\|_2$.

Step 4. Integrating the resulting equation from $-\nabla(\triangle \operatorname{div} q) \cdot \nabla \operatorname{div}(1.6)_2$ over $\mathbb{R}^n$, then using divergence theorem and the Cauchy inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\nabla(\triangle \operatorname{div} q)\right|^2 dx + \int_{\mathbb{R}^n} |\triangle \operatorname{div} q|^2\,dx &= \int_{\mathbb{R}^n} \nabla(\triangle \operatorname{div} q) \cdot \nabla(\triangle u)\,dx\\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla(\triangle \operatorname{div} q)\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla(\triangle u)\right|^2 dx,\end{aligned}$$

which implies by (2.1.27) and (2.1.28) $\|q(t)\|_4 \leqslant C\|u_0\|_3$. This completes the proof of Lemma 2.3. $\square$

Combination of Lemmas 2.1–2.3 proves Theorem 1.1. What's more, by the Gagliardo–Nirenberg inequality, we can easily show that (2.1.2) is always true provided $\|u_0\|_3$ is sufficiently small.

Now we turn to prove Theorem 1.2.

Proof of Theorem 1.2. First, as in [12,14], $u(x,t)$ satisfies the estimate $|u(t)|_1 \leqslant |u_0|_1$ and the proof can be found in [5]. The rest proof is divided into four steps.

Step 1. Multiplying (2.1.6) by $(1+t)^\alpha$, then integrating the resulting equation over $(0,t)$, we have

$$\begin{aligned}&(1+t)^\alpha \int_{\mathbb{R}^n} \left(u^2 + |\nabla u|^2\right) dx + 2\int_0^t \int_{\mathbb{R}^n} (1+\tau)^\alpha |\nabla u|^2\,dx\,d\tau\\ &\quad = \|u_0\|_1^2 + \alpha \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\alpha-1} u^2\,dx\,d\tau + \alpha \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\alpha-1} |\nabla u|^2\,dx\,d\tau + \int_0^t (1+\tau)^\alpha I_1\,d\tau\\ &\quad = \|u_0\|_1^2 + I_9 + I_{10} + \int_0^t (1+\tau)^\alpha I_1\,d\tau. \qquad (2.1.31)\end{aligned}$$

First, by the Gagliardo–Nirenberg and Young inequalities, we get for $\alpha > \frac{n}{2}$

$$\begin{aligned} I_9 &\leqslant C \int_0^t (1+\tau)^{\alpha-1} \left|\nabla u(\tau)\right|_2^{\frac{2n}{2+n}} \left|u(\tau)\right|_1^{\frac{4}{2+n}} d\tau \\ &\leqslant \frac{1}{4} \int_0^t (1+\tau)^{\alpha} \left|\nabla u(\tau)\right|_2^2 d\tau + C \int_0^t (1+\tau)^{\alpha-1-\frac{n}{2}} \left|u(\tau)\right|_1^2 d\tau \\ &\leqslant \frac{1}{4} \int_0^t (1+\tau)^{\alpha} \left|\nabla u(\tau)\right|_2^2 d\tau + C|u_0|_1^2 (1+t)^{\alpha-\frac{n}{2}}. \end{aligned} \tag{2.1.32}$$

Secondly, for sufficiently large $t > 0$, we choose $T \leqslant t$ such that $\alpha(1+T)^{-1} = \frac{1}{2}$ and divide the integral I_{10} into two parts corresponding to $[0, T]$ and $[T, t]$ respectively. Then we obtain

$$\alpha \int_0^T \int_{\mathbb{R}^n} (1+\tau)^{\alpha-1} |\nabla u|^2 \, dx \, d\tau \leqslant \alpha(1+T)^{\alpha-1} \int_0^t \left|\nabla u(\tau)\right|_2^2 d\tau \leqslant C\|u_0\|_1^2 \tag{2.1.33}$$

and

$$\begin{aligned} \alpha \int_T^t \int_{\mathbb{R}^n} (1+\tau)^{\alpha-1} |\nabla u|^2 \, dx \, d\tau &\leqslant \alpha(1+T)^{-1} \int_0^t (1+\tau)^{\alpha} \left|\nabla u(\tau)\right|_2^2 d\tau \\ &= \frac{1}{2} \int_0^t (1+\tau)^{\alpha} \left|\nabla u(\tau)\right|_2^2 d\tau. \end{aligned} \tag{2.1.34}$$

(2.1.33) and (2.1.34) show that

$$I_{10} \leqslant \frac{1}{2} \int_0^t (1+\tau)^{\alpha} \left|\nabla u(\tau)\right|_2^2 d\tau + C\|u_0\|_1^2. \tag{2.1.35}$$

Finally, from (2.1.2) we have

$$\int_0^t (1+\tau)^{\alpha} I_1 \, d\tau \leqslant C\varepsilon_1 \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\alpha} |\nabla u|^2 \, dx \, d\tau. \tag{2.1.36}$$

Substituting (2.1.32), (2.1.35) and (2.1.36) into (2.1.31), it follows that for $\alpha > \frac{n}{2}$

$$\begin{aligned} &(1+t)^{\alpha} \int_{\mathbb{R}^n} \left(u^2 + |\nabla u|^2\right) dx + \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\alpha} |\nabla u|^2 \, dx \, d\tau \\ &\quad \leqslant C\|u_0\|_1^2 + C|u_0|_1^2 (1+t)^{\alpha-\frac{n}{2}}. \end{aligned} \tag{2.1.37}$$

Since $\alpha > \frac{n}{2}$ in (2.1.37), we get the following decay rate

$$\big\|u(t)\big\|_1 \leqslant C\big(|u_0|_1 + \|u_0\|_1\big)(1+t)^{-\frac{n}{4}}. \tag{2.1.38}$$

Step 2. Multiplying (2.1.11) by $(1+t)^{\alpha+1}$, then integrating the resulting equation over $(0,t)$ yields

$$\begin{aligned}(1+t)^{\alpha+1}\int_{\mathbb{R}^n}\big(|\nabla u|^2 + |\triangle u|^2\big)\,dx + 2\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+1}|\triangle u|^2\,dx\,d\tau \\ = \|\nabla u_0\|_1^2 + I_{11} + I_{12} + \sum_{i=1}^{3}\int_0^t(1+\tau)^{\alpha+1}I_i\,d\tau,\end{aligned} \tag{2.1.39}$$

where

$$I_{11} = (\alpha+1)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha}|\nabla u|^2\,dx\,d\tau, \qquad I_{12} = (\alpha+1)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha}|\triangle u|^2\,dx\,d\tau.$$

First, we have from (2.1.37) that

$$I_{11} \leqslant C\|u_0\|_1^2 + C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}. \tag{2.1.40}$$

Also, in a similar manner to (2.1.35), we get

$$I_{12} \leqslant \frac{1}{4}\int_0^t(1+\tau)^{\alpha+1}\big|\triangle u(\tau)\big|_2^2\,d\tau + C\|u_0\|_2^2. \tag{2.1.41}$$

By integration-by-parts, the Gagliardo–Nirenberg inequality and (2.1.37)–(2.1.38), we obtain

$$\begin{aligned}\int_0^t(1+\tau)^{\alpha+1}I_1\,d\tau &= 2\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+1}u\nabla u\cdot\nabla u_{x_i}\,dx\,d\tau \\ &\leqslant \frac{1}{4}\int_0^t(1+\tau)^{\alpha+1}\big|\triangle u(\tau)\big|_2^2\,d\tau + C\|u_0\|_2^2 + C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}.\end{aligned} \tag{2.1.42}$$

One easily gets by (2.1.2)

$$\sum_{i=2}^{3}\int_0^t(1+\tau)^{\alpha+1}I_i\,d\tau \leqslant C\varepsilon_1\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+1}|\triangle u|^2\,dx\,d\tau. \tag{2.1.43}$$

Substituting (2.1.40)–(2.1.43) into (2.1.39), it follows that for $\alpha > \frac{n}{2}$

$$(1+t)^{\alpha+1}\int_{\mathbb{R}^n}\left(|\nabla u|^2+|\triangle u|^2\right)dx+\int_0^t(1+\tau)^{\alpha+1}\left|\triangle u(\tau)\right|_2^2\,d\tau$$
$$\leqslant C\|u_0\|_2^2+C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}. \tag{2.1.44}$$

Since $\alpha>\frac{n}{2}$ in (2.1.44), we get the following decay rate by (2.1.14)

$$\left\|\nabla u(t)\right\|_1\leqslant C\left(|u_0|_1+\|u_0\|_2\right)(1+t)^{-\frac{n}{4}-\frac{1}{2}}. \tag{2.1.45}$$

Step 3. Multiplying (2.1.23) by $(1+t)^{\alpha+2}$, then integrating the resulting equation over $(0,t)$ yields

$$(1+t)^{\alpha+2}\int_{\mathbb{R}^n}\left(|\triangle u|^2+\left|\nabla(\triangle u)\right|^2\right)dx+2\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+2}\left|\nabla(\triangle u)\right|^2dx\,d\tau$$
$$=\|\triangle u_0\|_1^2+I_{13}+I_{14}+\sum_{i=2}^{6}\int_0^t(1+\tau)^{\alpha+2}I_i\,d\tau, \tag{2.1.46}$$

where

$$I_{13}=(\alpha+2)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+1}|\triangle u|^2\,dx\,d\tau,$$
$$I_{14}=(\alpha+2)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+1}\left|\nabla(\triangle u)\right|^2dx\,d\tau.$$

From (2.1.44) we have that

$$I_{13}\leqslant C\|u_0\|_2^2+C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}. \tag{2.1.47}$$

Similar to the line of (2.1.35), we get from (2.1.27) and (2.1.28)

$$I_{14}\leqslant\frac{1}{4}\int_0^t(1+\tau)^{\alpha+2}\left|\nabla(\triangle u)(\tau)\right|_2^2\,d\tau+C\|u_0\|_3^2. \tag{2.1.48}$$

In order to estimate $\sum_{i=2}^3\int_0^t(1+\tau)^{\alpha+2}I_i\,d\tau$, we claim by (2.1.38) and (2.1.45)

$$\left|u(t)\right|_\infty^2\leqslant\begin{cases}C\left|u(t)\right|_2\left|u_{x_1}(t)\right|_2\leqslant C\left(|u_0|_1+\|u_0\|_2\right)^2(1+t)^{-1}, & n=1,\\ \left|u(t)\right|_2^{2-\frac{n}{2}}\left|\nabla^2u(t)\right|_2^{\frac{n}{2}}\leqslant C\left(|u_0|_1+\|u_0\|_2\right)^2(1+t)^{-\frac{3n}{4}}, & n=2,3.\end{cases} \tag{2.1.49}$$

By integration-by-parts and the Cauchy inequality, we obtain from (2.1.49)

$$\begin{aligned}
&\sum_{i=2}^{3}\int_0^t (1+\tau)^{\alpha+2} I_i\,d\tau \\
&\quad\leqslant \frac{1}{2}\int_0^t (1+\tau)^{\alpha+2}\big|\nabla(\triangle u)(\tau)\big|_2^2\,d\tau + C\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+2}|u|^2|\triangle u|^2\,dx\,d\tau \\
&\quad\leqslant \frac{1}{2}\int_0^t (1+\tau)^{\alpha+2}\big|\nabla(\triangle u)(\tau)\big|_2^2\,d\tau + C\|u_0\|_1^2 + C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}.
\end{aligned}\tag{2.1.50}$$

From (2.1.2), we have

$$\int_0^t (1+\tau)^{\alpha+2} I_4\,d\tau \leqslant C\varepsilon_1\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+2}\big|\nabla(\triangle u)\big|^2\,dx\,d\tau. \tag{2.1.51}$$

To estimate $\sum_{i=5}^{6}\int_0^t (1+\tau)^{\alpha+2} I_i\,d\tau$, it is sufficient to estimate the following terms:

$$C\int_0^t (1+\tau)^{\alpha+2}(I_7+I_8)\,d\tau \leqslant \left(\frac{1}{8}+C\varepsilon_1\right)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\alpha+2}\big|\nabla(\triangle u)\big|^2\,dx\,d\tau, \tag{2.1.52}$$

where we used (2.1.25) and (2.1.26).

Substituting (2.1.47), (2.1.48) and (2.1.50)–(2.1.52) into (2.1.46), we have for $\alpha > \frac{n}{2}$

$$\begin{aligned}
&(1+t)^{\alpha+2}\int_{\mathbb{R}^n}\Big(|\triangle u|^2 + \big|\nabla(\triangle u)\big|^2\Big) + \int_0^t (1+\tau)^{\alpha+2}\big|\nabla(\triangle u)(\tau)\big|_2^2\,d\tau \\
&\quad\leqslant C\|u_0\|_3^2 + C|u_0|_1^2(1+t)^{\alpha-\frac{n}{2}}.
\end{aligned}\tag{2.1.53}$$

Since $\alpha > \frac{n}{2}$ in (2.1.53), we get the following decay rate by (2.1.14) and (2.1.28)

$$\big\|\nabla^2 u(t)\big\|_1 \leqslant C\big(|u_0|_1 + \|u_0\|_3\big)(1+t)^{-\frac{n}{4}-1}. \tag{2.1.54}$$

Step 4. By Gagliardo–Nirenberg's inequality, (2.1.38) and (2.1.54), we have

$$\big|u(t)\big|_\infty \leqslant C\big|\nabla^2 u(t)\big|_2^{\frac{n}{4}}\big|u(t)\big|_2^{1-\frac{n}{4}} \leqslant C(1+t)^{-\frac{n}{2}} \quad \text{for } n=1,2,3. \tag{2.1.55}$$

By the Sobolev inequality, Gagliardo–Nirenberg inequality, (2.1.45) and (2.1.54), we have

$$\big|\nabla u(t)\big|_\infty \leqslant \begin{cases} C\big|u_{x_1}(t)\big|_2^{\frac{1}{2}}\big|u_{x_1x_1}(t)\big|_2^{\frac{1}{2}} \leqslant C(1+t)^{-1}, & n=1, \\ C\big|\nabla^3 u(t)\big|_2^{\frac{n}{4}}\big|\nabla u(t)\big|_2^{1-\frac{n}{4}} \leqslant C(1+t)^{-\frac{3n}{8}-\frac{1}{2}}, & n=2,3. \end{cases} \tag{2.1.56}$$

By using interpolation inequality, we have for $n=1,2,3$

$$\big|u(t)\big|_p \leqslant \begin{cases} C\big|u(t)\big|_2^{2(1-\frac{1}{p})}\big|u(t)\big|_1^{\frac{2}{p}-1} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 1 \leqslant p \leqslant 2, \\ C\big|u(t)\big|_\infty^{\frac{p-2}{p}}\big|u(t)\big|_2^{\frac{2}{p}} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 2 \leqslant p \leqslant \infty, \end{cases} \tag{2.1.57}$$

and for $2 \leqslant p \leqslant \infty$

$$\big|\nabla u(t)\big|_p \leqslant C\big|\nabla u(t)\big|_\infty^{\frac{p-2}{p}}\big|\nabla u(t)\big|_2^{\frac{2}{p}} \leqslant \begin{cases} C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, & n=1, \\ C(1+t)^{-\frac{(3n+4)p-2n}{8p}}, & n=2,3. \end{cases} \tag{2.1.58}$$

Combination of (2.1.57)–(2.1.58) shows (1.9). This completes the proof of Theorem 1.2. □

Remark 2.1. With the help of the decay estimates on u, we easily obtain the following decay estimates on $q(x,t)$ from the proof of Lemma 2.3

$$\begin{cases} \big\|q(t)\big\|_2 \leqslant C\big|\nabla u(t)\big|_2 \leqslant C(1+t)^{-\frac{n}{4}-\frac{1}{2}}, \\ \big\|\triangle \operatorname{div} q(t)\big\|_1 \leqslant C\big|\triangle u(t)\big|_2 \leqslant C(1+t)^{-\frac{n}{4}-1}. \end{cases}$$

2.2. The case of $3 < n < 8$

In this subsection, we will prove the global existence and obtain decay rates in the solution space $X_1(0,T)$ defined (2.1.1) for $3 < n < 8$ under the *a priori* assumption that $|u(t)|_{W^{1,\infty}}$ is sufficiently small. In fact, first, under the smallness assumption of $|u(t)|_{W^{1,\infty}}$, all decay estimates on u also hold for any $n > 3$. Thus, we need only to close the *a priori* assumption that $|u(t)|_{W^{1,\infty}}$ is sufficiently small. That is, we need to prove

$$\big|u(t)\big|_\infty \leqslant C\varepsilon_2, \qquad \big|\nabla u(t)\big|_\infty \leqslant C\varepsilon_2, \tag{2.2.1}$$

where $0 < \varepsilon_2 \ll 1$. This can be done by two methods. Either we need to obtain $H^l(\mathbb{R}^n)$-estimate for $l > [\frac{n}{2}]+1$, or we need to obtain $W^{3,p}(\mathbb{R}^n)$-estimate for $p > \frac{n}{2}$ by the Gagliardo–Nirenberg inequality. Since $H^l(\mathbb{R}^n)$-estimate on solutions is very complicated if n is slightly larger. Consequently, in the following, we will only give $W^{3,p}(\mathbb{R}^n)$-estimate for $p > \frac{n}{2}$. In addition, in the later proof, we require $(n-4)p < 2n$ $(n > 3)$, which and $p > \frac{n}{2}$ lead to $p = 4$ and $3 < n < 8$. To do this, besides $u_0 \in (L^1 \cap H^3)(\mathbb{R}^n)$, we need to furthermore assume $u_0 \in W^{3,4}(\mathbb{R}^n)$ and set the *a priori* assumption

$$\big\|u(t)\big\|_{W^{3,4}} \leqslant \varepsilon_2, \tag{2.2.2}$$

which implies (2.2.1) by the Gagliardo–Nirenberg inequality.

Now we give the proof of Theorem 1.3 under the *a priori* assumption (2.2.2).

Proof of Theorem 1.3. The proof is divided into five steps.

Step 1. Multiplying $(2.1.3)_1$ by $|u|^2u$, then integrating the resulting equation over $\mathbb{R}^n$, as in [5], we get by the Young inequality, Hausdorff–Young inequality and (2.1.38)

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\big|u(t)\big|_4^4 + \big|u(t)\big|_4^4 &\leqslant \int_{\mathbb{R}^n} |u|^3|\psi * u|\,dx \\ &\leqslant \frac{1}{2}\big|u(t)\big|_4^4 + C\big|(\psi * u)(t)\big|_4^4 \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\big|u(t)\big|_4^4 + C\big(\|\psi\|_{L^{\frac{4}{3}}}\big|u(t)\big|_2\big)^4 \\
&\leqslant \frac{1}{2}\big|u(t)\big|_4^4 + C\big(|u_0|_1 + \|u_0\|_1\big)^4 \|\psi\|_{L^{\frac{4}{3}}}^4 (1+t)^{-n},
\end{aligned}
\tag{2.2.3}
$$

which implies by Lemma 1.1

$$
\frac{d}{dt}\big|u(t)\big|_4^4 + \big|u(t)\big|_4^4 \leqslant C\big(|u_0|_1 + \|u_0\|_1\big)^4 (1+t)^{-n}. \tag{2.2.4}
$$

Multiplying (2.2.4) by e^t, then integrating the resulting equation over $(0,t)$, we obtain

$$
\big|u(t)\big|_4 \leqslant C\big(|u_0|_1 + \|u_0\|_1 + |u_0|_4\big)(1+t)^{-\frac{n}{4}}. \tag{2.2.5}
$$

Step 2. Differentiating $(2.1.3)_1$ with respect to x_j and multiplying the resulting equation by $|u_{x_j}|^2 u_{x_j}$, then integrating it with respect to x over $\mathbb{R}^n$, we get

$$
\begin{aligned}
\frac{1}{4}\frac{d}{dt}\big|u_{x_j}(t)\big|_4^4 + \big|u_{x_j}(t)\big|_4^4 &= -\frac{3}{4}\sum_{i=1}^n \int_{\mathbb{R}^n} u_{x_i}|u_{x_j}|^4\,dx + \int_{\mathbb{R}^n} |u_{x_j}|^2 u_{x_j}(\psi * u_{x_j})\,dx \\
&\leqslant C\varepsilon_2\big|u_{x_j}(t)\big|_4^4 + \frac{1}{2}\big|u_{x_j}(t)\big|_4^4 + C\big|u_{x_j}(t)\big|_2^4,
\end{aligned}
\tag{2.2.6}
$$

which implies

$$
\frac{d}{dt}\big|u_{x_j}(t)\big|_4^4 + \big|u_{x_j}(t)\big|_4^4 \leqslant C\big|u_{x_j}(t)\big|_2^4. \tag{2.2.7}
$$

Summing all estimates on (2.2.7) with respect to $j = 1, \dots, n$, we obtain by (2.1.45)

$$
\frac{d}{dt}\big|\nabla u(t)\big|_4^4 + \big|\nabla u(t)\big|_4^4 \leqslant C\big|\nabla u(t)\big|_2^4 \leqslant C\big(|u_0|_1 + \|u_0\|_2 + |u_0|_4\big)^4 (1+t)^{-n-2}. \tag{2.2.8}
$$

Using the same calculations as in (2.2.5), we obtain from (2.2.8)

$$
\big|\nabla u(t)\big|_4 \leqslant C\big(|u_0|_1 + \|u_0\|_2 + \|u_0\|_{W^{1,4}}\big)(1+t)^{-\frac{n}{4}-\frac{1}{2}}. \tag{2.2.9}
$$

Step 3. Differentiating $(2.1.3)_1$ twice with respect to x_1 and multiplying the resulting equation by $|u_{x_1x_1}|^2 u_{x_1x_1}$, then integrating it with respect to x over $\mathbb{R}^n$, we have

$$
\begin{aligned}
\frac{1}{4}\frac{d}{dt}\big|u_{x_1x_1}(t)\big|_4^4 + \big|u_{x_1x_1}(t)\big|_4^4 &= -\frac{3}{4}\sum_{i=1}^n \int_{\mathbb{R}^n} u_{x_i}|u_{x_1x_1}|^4\,dx - 2\sum_{i=1}^n \int_{\mathbb{R}^n} u_{x_1}|u_{x_1x_1}|^2 u_{x_1x_1}u_{x_1x_i}\,dx \\
&\quad + \int_{\mathbb{R}^n} |u_{x_1x_1}|^2 u_{x_1x_1}(\psi * u_{x_1x_1})\,dx \\
&\leqslant C\varepsilon_2\big|\nabla^2 u(t)\big|_4^4 + \frac{1}{2}\big|u_{x_1x_1}(t)\big|_4^4 + C\big|u_{x_1x_1}(t)\big|_2^4,
\end{aligned}
\tag{2.2.10}
$$

which implies

$$\frac{d}{dt}\left|u_{x_1x_1}(t)\right|_4^4+\left|u_{x_1x_1}(t)\right|_4^4\leqslant C\varepsilon_2\left|\nabla^2u(t)\right|_4^4+C\left|u_{x_1x_1}(t)\right|_2^4. \tag{2.2.11}$$

Similarly, we can also get the following estimates on $u_{x_ix_j}$ $(i,j=1,2,\dots,n)$

$$\frac{d}{dt}\left|u_{x_ix_j}(t)\right|_4^4+\left|u_{x_ix_j}(t)\right|_4^4\leqslant C\varepsilon_2\left|\nabla^2u(t)\right|_4^4+C\left|u_{x_ix_j}(t)\right|_2^4. \tag{2.2.12}$$

Summing on $i,j=1,2,\dots,n$ for the inequality (2.2.12), we have by (2.1.54)

$$\frac{d}{dt}\left|\nabla^2u(t)\right|_4^4+\left|\nabla^2u(t)\right|_4^4\leqslant C\left|\nabla^2u(t)\right|_2^4\leqslant C\big(|u_0|_1+\|u_0\|_2+|u_0|_4\big)^4(1+t)^{-n-4}. \tag{2.2.13}$$

Using the same calculations as in (2.2.5), we obtain form (2.2.13)

$$\left|\nabla^2u(t)\right|_4\leqslant C\big(|u_0|_1+\|u_0\|_2+\|u_0\|_{W^{2,4}}\big)(1+t)^{-\frac{n}{4}-1}. \tag{2.2.14}$$

Step 4. Differentiating $(2.1.3)_1$ three times with respect to x_1 and multiplying the resulting equation by $|u_{x_1x_1x_1}|^2u_{x_1x_1x_1}$, then integrating it with respect to x over $\mathbb{R}^n$, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left|u_{x_1x_1x_1}(t)\right|_4^4+\left|u_{x_1x_1x_1}(t)\right|_4^4\\
&=-\frac{3}{4}\sum_{i=1}^n\int_{\mathbb{R}^n}u_{x_i}|u_{x_1x_1x_1}|^4\,dx-3\sum_{i=1}^n\int_{\mathbb{R}^n}u_{x_1}|u_{x_1x_1x_1}|^2u_{x_1x_1x_1}u_{x_1x_1x_i}\,dx\\
&\quad-3\sum_{i=1}^n\int_{\mathbb{R}^n}u_{x_1x_1}u_{x_1x_i}|u_{x_1x_1x_1}|^2u_{x_1x_1x_1}\,dx+\int_{\mathbb{R}^n}|u_{x_1x_1x_1}|^2u_{x_1x_1x_1}(\psi*u_{x_1x_1x_1})\,dx\\
&=I_{15}+I_{16}+I_{17}+I_{18}.
\end{aligned} \tag{2.2.15}$$

Firstly, it easily follow from (2.2.1)

$$I_{15}\leqslant\frac{3}{4}n\left|\nabla u(t)\right|_\infty\left|u_{x_1x_1x_1}(t)\right|_4^4\leqslant C\varepsilon_2\left|u_{x_1x_1x_1}(t)\right|_4^4, \tag{2.2.16}$$

$$I_{16}\leqslant 3n\left|\nabla u(t)\right|_\infty\left|\nabla^3u(t)\right|_4^4\leqslant C\varepsilon_2\left|\nabla^3u(t)\right|_4^4. \tag{2.2.17}$$

Secondly, by using the Gagliardo–Nirenberg inequality and (2.2.2), we get

$$\begin{aligned}
I_{17}&\leqslant\frac{1}{4}\left|u_{x_1x_1x_1}(t)\right|_4^4+C\left|\nabla^2u(t)\right|_8^8\\
&\leqslant\frac{1}{4}\left|u_{x_1x_1x_1}(t)\right|_4^4+C\left|\nabla^3u(t)\right|_4^n\left|\nabla^2u(t)\right|_4^{8-n}\\
&\leqslant\frac{1}{4}\left|u_{x_1x_1x_1}(t)\right|_4^4+C\big(|u_0|_1+\|u_0\|_2+\|u_0\|_{W^{2,4}}\big)^{8-n}\left|\nabla^3u(t)\right|_4^4.
\end{aligned} \tag{2.2.18}$$

Finally, we have

$$I_{18}\leqslant\frac{1}{4}\left|u_{x_1x_1x_1}(t)\right|_4^4+C\left|u_{x_1x_1x_1}(t)\right|_2^4. \tag{2.2.19}$$

Substituting (2.2.16)–(2.2.19) into (2.2.15), we deduce

$$\begin{aligned}&\frac{d}{dt}\left|u_{x_1x_1x_1}(t)\right|_4^4+\left|u_{x_1x_1x_1}(t)\right|_4^4\\&\quad\leqslant C\big(\varepsilon_2+|u_0|_1+\|u_0\|_2+\|u_0\|_{W^{2,4}}\big)\left|\nabla^3u(t)\right|_4^4+C\left|u_{x_1x_1x_1}(t)\right|_2^4.\end{aligned}\tag{2.2.20}$$

We can also get similar estimates to (2.2.20) on $u_{x_ix_jx_k}$ $(i,j,k=1,2,\dots,n)$. Thus we have by (2.1.54)

$$\frac{d}{dt}\left|\nabla^3u(t)\right|_4^4+\left|\nabla^3u(t)\right|_4^4\leqslant C\left|\nabla^3u(t)\right|_2^4\leqslant C\big(|u_0|_1+\|u_0\|_3+\|u_0\|_{W^{2,4}}\big)^4(1+t)^{-n-4}.\tag{2.2.21}$$

Using the same calculations as in (2.2.5), we obtain from (2.2.21)

$$\left|\nabla^3u(t)\right|_4\leqslant C\big(|u_0|_1+\|u_0\|_3+\|u_0\|_{W^{3,4}}\big)(1+t)^{-\frac{n}{4}-1}.\tag{2.2.22}$$

Step 5. By the Gagliardo–Nirenberg inequality, we have from (2.2.5) and (2.2.14)

$$\left|u(t)\right|_\infty\leqslant C\left|\nabla^2u(t)\right|_4^{\frac{n}{8}}\left|u(t)\right|_4^{1-\frac{n}{8}}\leqslant C\big(|u_0|_1+\|u_0\|_3+\|u_0\|_{W^{3,4}}\big)(1+t)^{-\frac{3n}{8}}\tag{2.2.23}$$

and from (2.2.9) and (2.2.22)

$$\left|\nabla u(t)\right|_\infty\leqslant C\left|\nabla^3u(t)\right|_4^{\frac{n}{8}}\left|\nabla u(t)\right|_4^{1-\frac{n}{8}}\leqslant C\big(|u_0|_1+\|u_0\|_3+\|u_0\|_{W^{3,4}}\big)(1+t)^{-\frac{5n+8}{16}}.\tag{2.2.24}$$

By using interpolation inequality, we have

$$\left|u(t)\right|_p\leqslant\begin{cases}C\left|u(t)\right|_2^{2(1-\frac{1}{p})}\left|u(t)\right|_1^{\frac{2}{p}-1}\leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 1\leqslant p\leqslant 2,\\ C\left|u(t)\right|_\infty^{\frac{p-2}{p}}\left|u(t)\right|_2^{\frac{2}{p}}\leqslant C(1+t)^{-\frac{n(3p-2)}{8p}}, & 2\leqslant p\leqslant\infty,\end{cases}\tag{2.2.25}$$

$$\left|\nabla u(t)\right|_p\leqslant C\left|\nabla u(t)\right|_\infty^{\frac{p-2}{p}}\left|\nabla u(t)\right|_2^{\frac{2}{p}}\leqslant C(1+t)^{-\frac{(5n+8)p-2n}{16p}},\quad 2\leqslant p\leqslant\infty.\tag{2.2.26}$$

Combination of (2.2.25)–(2.2.26) shows (1.10). This completes the proof of Theorem 1.3. □

Finally, we have to show that the *a priori* assumption (2.2.2) holds. Since, under the *a priori* assumption (2.2.2), we have proved that (2.2.5), (2.2.9), (2.2.14) and (2.2.22) hold provided $0<\varepsilon_2\ll 1$. Therefore, (2.2.2) always true provided $|u_0|_1+\|u_0\|_3+\|u_0\|_{W^{3,4}}$ is sufficiently small.

2.3. Decay rates to diffusion waves

In order to obtain the decay rates to diffusion waves, we shall study the following Cauchy problem

$$\begin{cases}G_t-\triangle G_t-\triangle G=0, & x\in\mathbb{R}^n,\ t>0,\\ G(x,0)=\phi(x), & x\in\mathbb{R}^n.\end{cases}\tag{2.3.1}$$

By the Fourier transformation $\hat{G}(\xi,t)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}G(x,t)\,dx$, we have $\hat{G}(\xi,t)=\mathrm{e}^{-\frac{|\xi|^2}{1+|\xi|^2}t}\hat{\phi}(\xi)$. As in [8], we define the semigroup e^{tA} associated with the linearized system (2.3.1) by $G(x,t)=\mathrm{e}^{tA}\phi:=\mathcal{F}^{-1}(\mathrm{e}^{-\frac{|\xi|^2}{1+|\xi|^2}t}\hat{\phi}(\xi))$. We will show that the following decay estimate of the semigroup e^{tA}, which plays an essential role in the proof of Theorem 1.4.

Lemma 2.4. *Let $n \geqslant 1$ and $k \geqslant 0$ be integers and $p \in [1,2]$, $\phi \in (H^k \cap L^p)(\mathbb{R}^n)$. Then the semigroup e^{tA} for the linearized system* (2.3.1) *has the decay estimate*

$$\left|\nabla^k \mathrm{e}^{tA}\phi(t)\right|_2 \leqslant C\left\{(1+t)^{-\sigma(n,p,k)}|\phi|_p + \mathrm{e}^{-\frac{1}{2}(1+t)}\left|\nabla^k\phi\right|_2\right\} \tag{2.3.2}$$

for any $t \geqslant 0$, where $C = C(n,p,k)$ and $\sigma(n,p,k) = \frac{n}{2}(\frac{1}{p}-\frac{1}{2}) + \frac{k}{2}$.

Proof. By the Plancherel's theorem one has

$$\begin{aligned}
\left|\nabla^k \mathrm{e}^{tA}\phi(t)\right|_2^2 &\leqslant C\int_{\mathbb{R}^n}|\xi|^{2k}\mathrm{e}^{-\frac{2|\xi|^2}{1+|\xi|^2}t}\left|\hat{\phi}(\xi)\right|^2 d\xi \\
&= C\int_{|\xi|\leqslant 1}|\xi|^{2k}\mathrm{e}^{-\frac{2|\xi|^2}{1+|\xi|^2}t}\left|\hat{\phi}(\xi)\right|^2 d\xi + C\int_{|\xi|\geqslant 1}|\xi|^{2k}\mathrm{e}^{-\frac{2|\xi|^2}{1+|\xi|^2}t}\left|\hat{\phi}(\xi)\right|^2 d\xi \\
&= I_{19} + I_{20}.
\end{aligned} \tag{2.3.3}$$

For $1 \leqslant r \leqslant \infty$ and $\frac{1}{r} + \frac{1}{r'} = 1$, one easily claims

$$I_{19} \leqslant C(1+t)^{-\frac{n}{2r}-k}|\hat{\phi}|_{2r'}^2. \tag{2.3.4}$$

By the Fourier transformation inequality $|\hat{\phi}|_{2r'} \leqslant |\phi|_p$, $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{2r'} = 1$ (see [1]), we further rewrite (2.3.4) for $\frac{1}{2r} = \frac{1}{p} - \frac{1}{2}$ as follows

$$I_{19} \leqslant C(1+t)^{-\frac{n}{2r}-k}|\phi|_p^2. \tag{2.3.5}$$

In addition, I_{20} is bounded as follows:

$$I_{20} \leqslant C\mathrm{e}^{-t}\int_{|\xi|\geqslant 1}|\xi|^{2k}\left|\hat{\phi}(\xi)\right|^2 d\xi \leqslant C\mathrm{e}^{-(1+t)}\left|\nabla^k\phi\right|_2^2. \tag{2.3.6}$$

(2.3.5) and (2.3.6) show that (2.3.2) holds. This completes the proof of Lemma 2.4. □

Next, by simple calculations, we have following results.

Lemma 2.5. *Let β and γ be positive constants. Then*

$$\int_0^{\frac{t}{2}}(1+t-s)^{-\beta}(1+s)^{-\gamma}\,ds \leqslant \begin{cases} C(1+t)^{-\beta}, & \gamma > 1, \\ C(1+t)^{-\beta}\ln(1+t), & \gamma = 1, \\ C(1+t)^{-(\beta+\gamma-1)}, & \gamma < 1, \end{cases} \tag{2.3.7}$$

$$\int_{\frac{t}{2}}^{t}(1+t-s)^{-\beta}(1+s)^{-\gamma}\,ds \leqslant \begin{cases} C(1+t)^{-\gamma}, & \beta > 1, \\ C(1+t)^{-\gamma}\ln(1+t), & \beta = 1, \\ C(1+t)^{-(\beta+\gamma-1)}, & \beta < 1. \end{cases} \tag{2.3.8}$$

Now, we define the time-asymptotic profile $G = G(x,t)$ satisfying

$$\begin{cases} G_t - \triangle G_t - \triangle G = 0, & x \in \mathbb{R}^n,\ t > 0, \\ G(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.3.9}$$

Then, the perturbation $W = u - G$ satisfies

$$\begin{cases} W_t - \triangle W_t - \triangle W = \triangle\left(\sum_{i=1}^{n} uu_{x_i}\right) - \sum_{i=1}^{n} uu_{x_i}, \\ W(x,0) = W_0(x) = 0. \end{cases} \tag{2.3.10}$$

In what follows, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. We rewrite (2.3.10) into an integral form by Duhamel's principle:

$$\begin{aligned} W(x,t) &= \int_0^t e^{(t-s)A} \triangle\left(\sum_{i=1}^{n} uu_{x_i}\right)(s)\,ds - \int_0^t e^{(t-s)A} \sum_{i=1}^{n} (uu_{x_i})(s)\,ds \\ &= \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t}\right) e^{(t-s)A} \triangle\left(\sum_{i=1}^{n} uu_{x_i}\right)(s)\,ds - \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t}\right) e^{(t-s)A} \sum_{i=1}^{n} (uu_{x_i})(s)\,ds \\ &= \left(I_{21}^1 + I_{21}^2\right) - \left(I_{22}^1 + I_{22}^2\right). \end{aligned} \tag{2.3.11}$$

By integration-by-parts, Holder's inequality, Lemmas 2.4, 2.5 and Theorem 1.2, we have

$$\begin{aligned} \left|I_{21}^1\right|_2 + \left|I_{22}^1\right|_2 &\leqslant \frac{1}{2} \sum_{i=1}^{n} \int_0^{\frac{t}{2}} \left(\left|\partial_{x_i} \triangle e^{(t-s)A} u^2(s)\right|_2 + \left|\partial_{x_i} e^{(t-s)A} u^2(s)\right|_2\right) ds \\ &\leqslant C \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{1}{2})} \left|u^2(s)\right|_1 ds + C \int_0^{\frac{t}{2}} e^{-\frac{1}{2}(1+t-s)} \left(\left|\nabla^3 u^2(s)\right|_2 + \left|\nabla u^2(s)\right|_2\right) ds \\ &\leqslant C \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{1}{2})} (1+s)^{-\frac{n}{2}} ds + C \int_0^{\frac{t}{2}} e^{-\frac{1}{2}(1+t-s)} (1+s)^{-(\frac{3n}{4}+\frac{1}{2})} ds \\ &\leqslant \begin{cases} C(1+t)^{-\frac{1}{4}}, & n = 1, \\ C(1+t)^{-1} \ln(1+t), & n = 2, \\ C(1+t)^{-\frac{5}{4}}, & n = 3, \end{cases} \end{aligned} \tag{2.3.12}$$

$$\begin{aligned} \left|I_{21}^2\right|_2 &\leqslant \sum_{i,j=1}^{n} \int_{\frac{t}{2}}^{t} \left|\partial_{x_j} e^{(t-s)A} (uu_{x_i})_{x_j}(s)\right|_2 ds \\ &\leqslant \sum_{i,j=1}^{n} \int_{\frac{t}{2}}^{t} (1+t-s)^{-(\frac{n}{4}+\frac{1}{2})} \left|(uu_{x_i})_{x_j}(s)\right|_1 ds + \sum_{i,j=1}^{n} \int_{\frac{t}{2}}^{t} e^{-\frac{1}{2}(1+t-s)} \left|\nabla (uu_{x_i})_{x_j}(s)\right|_2 ds \\ &\leqslant C \int_{\frac{t}{2}}^{t} (1+t-s)^{-(\frac{n}{4}+\frac{1}{2})} (1+s)^{-(\frac{n}{2}+1)} ds + C \int_{\frac{t}{2}}^{t} e^{-\frac{1}{2}(1+t-s)} (1+s)^{-(\frac{3n}{4}+1)} ds, \end{aligned} \tag{2.3.13}$$

$$\begin{aligned}
\left|I_{22}^2\right|_2 &\leqslant \sum_{i=1}^n \int_{\frac{t}{2}}^t \left|e^{(t-s)A}(uu_{x_i})(s)\right|_2 ds \\
&\leqslant \sum_{i=1}^n \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{n}{4}} \left|uu_{x_i}(s)\right|_1 ds + \sum_{i=1}^n \int_{\frac{t}{2}}^t e^{-\frac{1}{2}(1+t-s)} \left|uu_{x_i}(s)\right|_2 ds \\
&\leqslant C \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{n}{4}} (1+s)^{-(\frac{n}{2}+\frac{1}{2})} ds + C \int_{\frac{t}{2}}^t e^{-\frac{1}{2}(1+t-s)} (1+s)^{-(\frac{3n}{4}+\frac{1}{2})} ds. \qquad (2.3.14)
\end{aligned}$$

(2.3.13) and (2.3.14) show

$$\left|I_{21}^2\right|_2 + \left|I_{22}^2\right|_2 \leqslant \begin{cases} C(1+t)^{-\frac{1}{4}}, & n=1, \\ C(1+t)^{-1}, & n=2, \\ C(1+t)^{-\frac{7}{4}}, & n=3. \end{cases} \qquad (2.3.15)$$

Substituting (2.3.12) and (2.3.15) into (2.3.11), we obtain

$$\left|W(t)\right|_2 \leqslant \begin{cases} C(1+t)^{-\frac{1}{4}}, & n=1, \\ C(1+t)^{-1}\ln(1+t), & n=2, \\ C(1+t)^{-\frac{5}{4}}, & n=3. \end{cases} \qquad (2.3.16)$$

Similar to the case of $n = 1, 2, 3$, for the case of $3 < n < 8$, we have the following result

$$\left|W(t)\right|_2 \leqslant C(1+t)^{-(\frac{n}{4}+\frac{1}{2})}. \qquad (2.3.17)$$

Combination of (2.3.16) and (2.3.17) shows (1.11). This completes the proof of Theorem 1.4. □

3. The case with different end states $u_- < u_+$

In this section, we consider the case with the different end states $u_- < u_+$, i.e., consider the following Cauchy problem

$$\begin{cases} u_t + \sum_{i=1}^n uu_{x_i} - \triangle u_t - \triangle\left(\sum_{i=1}^n uu_{x_i}\right) - \triangle u = 0, \\ u(x,0) = u_0(x) \to u_\pm, \quad x_1 \to \pm\infty, \end{cases} \qquad (3.0.1)$$

where $u_\pm$ are given constant states and $u_- < u_+$. We will study the convergence rates of the planar rarefaction wave for the Cauchy problem (3.0.1). First of all, as in [7,14,29], the rarefaction waves are defined by $u^R(x_1/t)$, which is the center rarefaction wave of the inviscid Burgers equation connecting the states u_- and u_+. That is, $u^R(x_1/t)$ is a continuous weak solution to the Riemann problem for the inviscid Burgers equation with Riemann initial data

$$\begin{cases} u_t^R + u^R u_{x_1}^R = 0, \\ u^R(x_1,0) = u_0^R(x_1) = \begin{cases} u_+, & x_1 > 0, \\ u_-, & x_1 < 0. \end{cases} \end{cases} \qquad (3.0.2)$$

As in [7,14,25], we construct smooth approximate rarefaction wave $U(x_1,t)$ satisfies for any fixed $t_0>0$

$$\begin{cases} U_t + UU_{x_1} = U_{x_1x_1}, \quad t>-t_0, \\ U(x_1,-t_0) = U_0^R(x_1). \end{cases} \tag{3.0.3}$$

The properties of the solution $U(x_1,t)$ to the Cauchy problem (3.0.3) can be found in [7,14].

3.1. The case of $n=1$

In this subsection, we denote the one-dimensional space invariant x_1 by x without confusion for ease of notation and consider the case for $n=1$ in (3.0.1):

$$\begin{cases} z_t + zz_x - z_{xxt} - (zz_x)_{xx} - z_{xx} = 0, \\ z(x,0) = z_0(x) \to u_\pm, \quad x\to\pm\infty. \end{cases} \tag{3.1.1}$$

Let

$$z(x,t) = w(x,t) + U(x,t). \tag{3.1.2}$$

Then (3.1.1) can be reformulated as

$$\begin{cases} w_t + (Uw)_x + ww_x - w_{xxt} - (Uw)_{xxx} - (ww_x)_{xx} - w_{xx} = \partial_x^4 U, \\ w_0(x) = z_0(x) - U_0(x). \end{cases} \tag{3.1.3}$$

We seek the solution of (3.1.3) in the solution space $X_2(0,T)$ defined by

$$X_2(0,T) = \left\{ w\in C^0\big([0,T];H^4(\mathbb{R})\big),\ w_x\in L^2\big([0,T];H^3(\mathbb{R})\big) \right\}$$

for $0<T\leqslant\infty$ under the *a priori* assumption

$$\big\| w(t)\big\|_4 \leqslant \varepsilon_3, \quad 0\leqslant t<\infty, \tag{3.1.4}$$

which means that by the Sobolev inequality

$$\big|w(t)\big|_\infty \leqslant C\varepsilon_3, \qquad \big|w_x(t)\big|_\infty \leqslant C\varepsilon_3, \qquad \big|w_{xx}(t)\big|_\infty \leqslant C\varepsilon_3, \qquad \big|\partial_x^3 w(t)\big|_\infty \leqslant C\varepsilon_3, \tag{3.1.5}$$

where $0<\varepsilon_3\ll 1$.

First our aim is to prove the decay rates on w stated in the following lemma.

Lemma 3.1. *Assume that $w_0\in (L^1\cap H^4)(\mathbb{R})$ and that $|w_0|_1 + \|w_0\|_4 + \delta_0$ is suitably small. Then the global solution $w(x,t)$ satisfies the decay estimates for $t\geqslant 0$*

$$\begin{cases} \big\|\partial_x^k w(t)\big\|_1 \leqslant C\big(\|w_0\|_{k+1} + M_k\big)(1+t)^{-(\frac14+\frac k2)}, \quad k=0,1,2,3, \\ \big|\partial_x^k w(t)\big|_\infty \leqslant C\big(\|w_0\|_{k+1}+M_k\big)^{\frac12}\big(\|w_0\|_{k+2}+M_{k+1}\big)^{\frac12}(1+t)^{-\frac12(k+1)}, \quad k=0,1,2, \\ \big|\partial_x^3 w(t)\big|_\infty \leqslant C\big(\|w_0\|_4 + M_3\big)(1+t)^{-\frac74}, \end{cases} \tag{3.1.6}$$

where $M_k = |w_0|_1(1+|w_0|_1)^{2k} + \delta_0$.

Proof. The L^2-estimates on w, w_x and w_{xx} have been derived in [14]. Although the proof of the L^2-estimates on $\partial_x^3 w$ and $\partial_x^4 w$ is similar to that of w, w_x and w_{xx}, for the completeness and reader's convenience, here we give an outline of the proof of the L^2-estimates on $\partial_x^3 w$ and $\partial_x^4 w$. To this end, differentiating $(3.1.3)_1$ three times with respect to x and multiplying $2\partial_x^3 w$, then integrating the resulting equation over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\left\{\left|\partial_x^3 w(t)\right|_2^2+\left|\partial_x^4 w(t)\right|_2^2\right\}_t+2\left|\partial_x^4 w(\tau)\right|_2^2+\int_{\mathbb{R}}\left\{7U_x\left(\partial_x^3 w\right)^2+9U_x\left(\partial_x^4 w\right)^2\right\}dx\\
&\quad=\int_{\mathbb{R}}\left\{2\partial_x^3 w\partial_x^7 U-2w\partial_x^3 w\partial_x^4 U-8\partial_x w\partial_x^3 w\partial_x^3 U-8w_x\left(\partial_x^3 w\right)^2\right.\\
&\qquad-12U_{xx}w_{xx}\partial_x^3 w-2w\partial_x^3 w\partial_x^4 w-6w_{xx}^2\partial_x^3 w-2w\partial_x^4 w\partial_x^5 U-10w_x\partial_x^4 w\partial_x^4 U\\
&\qquad\left.-20w_{xx}\partial_x^4 w\partial_x^3 U-20U_{xx}\partial_x^3 w\partial_x^4 w-9w_x\left(\partial_x^4 w\right)^2+10\left(\partial_x^3 w\right)^3\right\}dx. \qquad (3.1.7)
\end{aligned}$$

By using the Cauchy–Schwartz inequality, (3.1.4), (3.1.7) and tedious calculations, we arrive at

$$\begin{aligned}
&\left|\partial_x^3 w(t)\right|_2^2+\left|\partial_x^4 w(t)\right|_2^2+\int_0^t\left|\partial_x^4 w(\tau)\right|_2^2 d\tau+\int_0^t\int_{\mathbb{R}}\left\{U_x\left(\partial_x^3 w\right)^2+U_x\left(\partial_x^4 w\right)^2\right\}dx\,d\tau\\
&\quad\leqslant C\left(\left|\partial_x^3 w_0\right|_2^2+\left|\partial_x^4 w_0\right|_2^2+\delta_0^2\right). \qquad (3.1.8)
\end{aligned}$$

Multiplying (3.1.7) by $(1+t)^{\nu+3}$ for any fixed $\nu>3$, then integrating the resulting equation over $(0,t)$, we have

$$\begin{aligned}
&(1+t)^{\nu+3}\left\|\partial_x^3 w(t)\right\|_1^2+\int_0^t(1+\tau)^{\nu+3}\left\{2\left|\partial_x^4 w(\tau)\right|_2^2+7\left|\sqrt{U_x}\partial_x^3 w(\tau)\right|_2^2+9\left|\sqrt{U_x}\partial_x^4 w(\tau)\right|_2^2\right\}d\tau\\
&\quad=\left\|\partial_x^3 w_0\right\|_1^2+J_1+J_2+J_3, \qquad (3.1.9)
\end{aligned}$$

where

$$J_1=(\nu+3)\int_0^t(1+\tau)^{\nu+2}\left|\partial_x^3 w(\tau)\right|_2^2 d\tau,\qquad J_2=(\nu+3)\int_0^t(1+\tau)^{\nu+2}\left|\partial_x^4 w(\tau)\right|_2^2 d\tau,$$

$$\begin{aligned}
J_3=&\int_0^t\int_{\mathbb{R}}(1+\tau)^{\nu+3}\left\{2\partial_x^3 w\partial_x^7 U-2w\partial_x^3 w\partial_x^4 U-8\partial_x w\partial_x^3 w\partial_x^3 U-8w_x\left(\partial_x^3 w\right)^2\right.\\
&-12U_{xx}w_{xx}\partial_x^3 w-2w\partial_x^3 w\partial_x^4 w-6w_{xx}^2\partial_x^3 w-2w\partial_x^4 w\partial_x^5 U-10w_x\partial_x^4 w\partial_x^4 U\\
&\left.-20w_{xx}\partial_x^4 w\partial_x^3 U-20U_{xx}\partial_x^3 w\partial_x^4 w-9w_x\left(\partial_x^4 w\right)^2+10\left(\partial_x^3 w\right)^3\right\}dx\,d\tau.
\end{aligned}$$

First, we can estimate J_1 by the decay estimates on $\|\partial_x^2 w(t)\|_1$ in [14]

$$J_1\leqslant C\|w_0\|_3^2+CM_2^2(1+t)^{\nu-\frac{1}{2}}. \qquad (3.1.10)$$

Secondly, by (3.1.8) J_2 can be bounded as follows:

$$J_2 \leqslant \frac{1}{2}\int_0^t (1+\tau)^{\nu+3}\left|\partial_x^4 w(\tau)\right|_2^2 d\tau + C\left(\|w_0\|_4^2 + \delta_0^2\right). \tag{3.1.11}$$

Finally, by tedious calculations, J_3 can be estimated as follows:

$$J_3 \leqslant \frac{1}{2}\int_0^t (1+\tau)^{\nu+3}\left|\partial_x^4 w(\tau)\right|_2^2 d\tau + C\|w_0\|_4^2 + CM_3^2(1+t)^{\nu-\frac{1}{2}}. \tag{3.1.12}$$

Substituting (3.1.10)–(3.1.12) into (3.1.9), together with the results in [14], we obtain $(3.1.6)_1$. By the Sobolev inequality and $(3.1.6)_1$, one easily gets $(3.1.6)_2$ and $(3.1.6)_3$. This completes the proof of Lemma 3.1. □

By (3.1.2), Lemma 3.1 and the properties on $U(x,t)$, we have the following estimates which will be used in the next section.

Lemma 3.2. *The solution $z(x,t)$ to the Cauchy problem* (3.1.1) *satisfies*

$$\left|\partial_x^k z(t)\right|_2 \leqslant C(1+t)^{-(\frac{1}{4}+\frac{k}{2})}, \quad k=2,3, \tag{3.1.13}$$

$$\begin{cases} \left|\partial_x z(t)\right|_2 \leqslant C\left(\|w_0\|_2 + M_1 + \delta_0^{\frac{1}{2}}\right)(1+t)^{-\frac{1}{2}}, \\ \left|\partial_x^k z(t)\right|_2 \leqslant C\left(\|w_0\|_{k+1} + M_k + \delta_0\right)(1+t)^{-\frac{1}{2}(k-\frac{1}{2})}, \quad k=2,3, \\ \left|\partial_x^4 z(t)\right|_2 \leqslant C\left(\|w_0\|_4 + M_3 + \delta_0\right)(1+t)^{-\frac{7}{4}}, \end{cases} \tag{3.1.14}$$

$$\begin{cases} \left|\partial_x z(t)\right|_\infty \leqslant C(1+t)^{-1}, \\ \left|\partial_x^2 z(t)\right|_\infty \leqslant C(1+t)^{-\frac{3}{2}}, \end{cases} \tag{3.1.15}$$

$$\begin{cases} \left|\partial_x z(t)\right|_\infty \leqslant C\left(\|w_0\|_2 + M_1 + \delta_0^{\frac{1}{2}}\right)^{\frac{1}{2}}(1+t)^{-\frac{7}{8}}, \\ \left|\partial_x^2 z(t)\right|_\infty \leqslant C\left(\|w_0\|_3 + M_2 + \delta_0\right)^{\frac{1}{2}}(1+t)^{-\frac{5}{4}}, \\ \left|\partial_x^3 z(t)\right|_\infty \leqslant C\left(\|w_0\|_4 + M_3 + \delta_0\right)^{\frac{1}{2}}(1+t)^{-\frac{7}{4}}. \end{cases} \tag{3.1.16}$$

As expected, if we assume that $z_0'(x) > 0$ for $x \in \mathbb{R}$, then the solution $z(x,t)$ of (3.1.1) *is a strictly increasing function of $x \in \mathbb{R}$, which will play an important role in next subsection. As in* [4], *this can be shown by using the maximum principle, cf.* [28].

Lemma 3.3. *Suppose that $z_0(x)$ is monotonically increasing, i.e., $z_0'(x) > 0$ for $x \in \mathbb{R}$. Then the solution $z(x,t)$ of* (3.1.1) *satisfies $\frac{\partial}{\partial x}z(x,t) > 0$, $(x,t) \in \mathbb{R}\times[0,\infty)$.*

3.2. The case of $n = 2, 3$

In this subsection, as in [4,12,20,25,27], to consider the n-dimensional ($n = 2$ or 3) Cauchy problem corresponding to Cauchy problem (3.0.1), we introduce $v(x,t)$ as the perturbation from the planar wave $z(x_1,t)$ and set

$$u(x,t) = z(x_1,t) + v(x,t). \tag{3.2.1}$$

Then, when $n = 2, 3$, the Cauchy problem (3.0.1) is transformed to

$$\begin{cases} v_t + v\sum_{i=1}^{n} v_{x_i} - \triangle v_t - \triangle\left(v\sum_{i=1}^{n} v_{x_i}\right) - \triangle v + \sum_{i=1}^{n} (zv)_{x_i} - \triangle\left(\sum_{i=1}^{n}(zv)_{x_i}\right) = 0, \\ v(x,0) \equiv v_0(x) = u_0(x) - z_0(x_1) \to 0, \quad x_1 \to \pm\infty. \end{cases} \tag{3.2.2}$$

Now we give the proof of Theorem 1.5 under the *a priori* assumptions

$$\left|v(t)\right|_\infty \leqslant C\varepsilon_4, \qquad \left|\nabla v(t)\right|_\infty \leqslant C\varepsilon_4, \quad 0 < \varepsilon_4 \ll 1. \tag{3.2.3}$$

Proof of Theorem 1.5. First, $v(x,t)$ satisfies the estimate $|v(t)|_1 \leqslant |v_0|_1$ and the proof can be found in [5]. In the rest proof, we only devoted ourself to estimating the terms involving z or derivatives of z. The other terms $\bar{J}_i$ $(i = 1, \dots, 6)$ are estimated in the same computations as those in Section 2.

Step 1. Multiplying $(3.2.2)_1$ by $2v$, we have

$$\begin{aligned} &\left(v^2 + |\nabla v|^2\right)_t + z_{x_1} v^2 + 2|\nabla v|^2 - 2v\triangle\left(v\sum_{i=1}^{n} v_{x_i}\right) - 2v\triangle\left(\sum_{i=1}^{n}(zv)_{x_i}\right) \\ &\quad - 2\,\mathrm{div}\left\{v(\nabla v + \nabla v_t)\right\} + \sum_{i=1}^{n}\left(\frac{2}{3}v^3 + zv^2\right)_{x_i} = 0. \end{aligned} \tag{3.2.4}$$

Rewrite the fifth term on the left-hand side of (3.2.4) as follows:

$$\begin{aligned} -2v\triangle\left(\sum_{i=1}^{n}(zv)_{x_i}\right) &= -2\,\mathrm{div}\left\{v\nabla\left(\sum_{i=1}^{n}(zv)_{x_i}\right)\right\} + 2\nabla v\cdot\nabla\left(\sum_{i=1}^{n}(zv)_{x_i}\right) \\ &= -2\,\mathrm{div}\left\{v\nabla\left(\sum_{i=1}^{n}(zv)_{x_i}\right)\right\} + 2z_{x_1}v_{x_1}\sum_{i=1}^{n} v_{x_i} + 2z_{x_1x_1} v v_{x_1} \\ &\quad + z_{x_1}|\nabla v|^2 + \sum_{i=1}^{n}\left(z|\nabla v|^2\right)_{x_i}. \end{aligned} \tag{3.2.5}$$

Substituting (3.2.5) into (3.2.4) and integrating the resulting equation over $\mathbb{R}^n$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n}\left(v^2 + |\nabla v|^2\right)dx + \int_{\mathbb{R}^n} z_{x_1}\left(v^2 + |\nabla v|^2\right)dx + 2\int_{\mathbb{R}^n}|\nabla v|^2\,dx \\ &= -\int_{\mathbb{R}^n}|\nabla v|^2\sum_{i=1}^{n} v_{x_i}\,dx - 2\int_{\mathbb{R}^n} z_{x_1}v_{x_1}\sum_{i=1}^{n} v_{x_i}\,dx - 2\int_{\mathbb{R}^n} z_{x_1x_1} v v_{x_1}\,dx = \bar{J}_1 + \sum_{i=4}^{5} J_i. \end{aligned} \tag{3.2.6}$$

First, we have

$$J_4 \leqslant C\left|z_{x_1}(t)\right|_\infty \int_{\mathbb{R}^n}|\nabla v|^2\,dx. \tag{3.2.7}$$

Secondly, by the Gagliardo–Nirenberg and Young inequalities, we get

$$
\begin{aligned}
J_5 &\leqslant \frac{1}{4}\int\limits_{\mathbb{R}^n} v_{x_1}^2\,dx + C\big|z_{x_1x_1}(t)\big|_\infty^2\big|v(t)\big|_2^2 \\
&\leqslant \frac{1}{4}\int\limits_{\mathbb{R}^n} |\nabla v|^2\,dx + C\big|z_{x_1x_1}(t)\big|_\infty^2\big|\nabla v(t)\big|_2^{\frac{2n}{2+n}}\big|v(t)\big|_1^{\frac{4}{2+n}} \\
&\leqslant \frac{1}{2}\int\limits_{\mathbb{R}^n} |\nabla v|^2\,dx + C\big|z_{x_1x_1}(t)\big|_\infty^{2(1+\frac{n}{2})}|v_0|_1^2. \qquad (3.2.8)
\end{aligned}
$$

Substituting (3.2.7) and (3.2.8) into (3.2.6), then integrating the resulting equation over $(0,t)$, and using the estimates on $\bar{J}_1$, we obtain

$$
\begin{aligned}
&\big\|v(t)\big\|_1^2 + \int\limits_0^t\int\limits_{\mathbb{R}^n}\big\{|\nabla v|^2 + z_{x_1}\big(v^2+|\nabla v|^2\big)\big\}\,dx\,d\tau \\
&\quad\leqslant C\big(\|v_0\|_1^2 + \|w_0\|_3^2 + M_2^2 + \delta_0^2\big). \qquad (3.2.9)
\end{aligned}
$$

Multiplying (3.2.6) by $(1+t)^\beta$, then integrating the resulting equation over $(0,t)$, we have

$$
\begin{aligned}
&(1+t)^\beta\int\limits_{\mathbb{R}^n}\big(v^2+|\nabla v|^2\big)\,dx + \int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^\beta\big\{2|\nabla v|^2 + z_{x_1}\big(v^2+|\nabla v|^2\big)\big\}\,dx\,d\tau \\
&\quad= \|v_0\|_1^2 + \int\limits_0^t(1+\tau)^\beta J_4\,d\tau + \int\limits_0^t(1+\tau)^\beta J_5\,d\tau + \bar{J}_2, \qquad (3.2.10)
\end{aligned}
$$

where

$$
\bar{J}_2 = \beta\int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^{\beta-1}\big(v^2+|\nabla v|^2\big)\,dx\,d\tau - \int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^\beta|\nabla v|^2\sum_{i=1}^n v_{x_i}\,dx\,d\tau.
$$

By (3.2.3) and (3.1.15), we obtain

$$
\int\limits_0^t(1+\tau)^\beta J_4\,d\tau \leqslant C\big|z_{x_1}(t)\big|_\infty\int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^\beta|\nabla v|^2\,dx\,d\tau, \qquad (3.2.11)
$$

$$
\begin{aligned}
\int\limits_0^t(1+\tau)^\beta J_5\,d\tau &\leqslant \frac{1}{4}\int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^\beta v_{x_1}^2\,dx\,d\tau + C\int\limits_0^t(1+\tau)^\beta\big|z_{x_1x_1}(\tau)\big|_\infty^2\int\limits_{\mathbb{R}^n}v^2\,dx\,d\tau \\
&\leqslant \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^\beta|\nabla v|^2\,dx\,d\tau + C(1+t)^{\beta-\frac{n}{2}}. \qquad (3.2.12)
\end{aligned}
$$

Substituting (3.2.11) and (3.2.12) into (3.2.10), using the estimate on $\bar{J}_2$, we have for $\beta > \frac{n}{2}$

$$(1+t)^{\beta}\int\limits_{\mathbb{R}^n}\big(v^2+|\nabla v|^2\big)\,dx+\int\limits_0^t\int\limits_{\mathbb{R}^n}(1+\tau)^{\beta}\big\{|\nabla v|^2+z_{x_1}\big(v^2+|\nabla v|^2\big)\big\}\,dx\,d\tau$$
$$\leqslant C\big(1+(1+t)^{\beta-\frac{n}{2}}\big). \tag{3.2.13}$$

Consequently, since $\beta>\frac{n}{2}$ in (3.2.13) we get

$$\big\|v(t)\big\|_1\leqslant C(1+t)^{-\frac{n}{4}}. \tag{3.2.14}$$

Step 2. Multiplying $(3.2.2)_1$ by $-2\triangle v$, similar to the calculations of (3.2.5), we have

$$\begin{aligned}&\big(|\nabla v|^2+|\triangle v|^2\big)_t+2|\triangle v|^2+z_{x_1}|\nabla v|^2+|\nabla v|^2\sum_{i=1}^n v_{x_i}+2z_{x_1}v_{x_1}\sum_{i=1}^n v_{x_i}+2z_{x_1x_1}vv_{x_1}\\&\quad+2\triangle v\triangle\left(v\sum_{i=1}^n v_{x_i}\right)+2\triangle v\triangle\left(\sum_{i=1}^n (zv)_{x_i}\right)\\&\quad-2\,\mathrm{div}\left\{\left(v_t+\sum_{i=1}^n vv_{x_i}+\sum_{i=1}^n (zv)_{x_i}\right)\nabla v\right\}+\sum_{i=1}^n\big\{(z+v)|\nabla v|^2\big\}_{x_i}=0.\end{aligned} \tag{3.2.15}$$

Rewrite the eighth term on the left-hand side of (3.2.15) as follows:

$$\begin{aligned}2\triangle v\triangle\left(\sum_{i=1}^n (zv)_{x_i}\right)&=z_{x_1}|\triangle v|^2+2z_{x_1x_1}\sum_{i=1}^n v_{x_i}\triangle v+4z_{x_1}\sum_{i=1}^n v_{x_1x_i}\triangle v\\&\quad+4z_{x_1x_1}v_{x_1}\triangle v+2z_{x_1x_1x_1}v\triangle v+\sum_{i=1}^n\big\{z|\triangle v|^2\big\}_{x_i}.\end{aligned} \tag{3.2.16}$$

Substituting (3.2.16) into (3.2.15), then integrating the resulting equation over $\mathbb{R}^n$, we have

$$\frac{d}{dt}\int\limits_{\mathbb{R}^n}\big(|\nabla v|^2+|\triangle v|^2\big)\,dx+\int\limits_{\mathbb{R}^n}\big\{2|\triangle v|^2+z_{x_1}\big(|\nabla v|^2+|\triangle v|^2\big)\big\}\,dx=\sum_{i=4}^9 J_i+\bar{J}_3, \tag{3.2.17}$$

where

$$\begin{cases}J_6=-4\displaystyle\int\limits_{\mathbb{R}^n}z_{x_1}\sum_{i=1}^n v_{x_1x_i}\triangle v\,dx, & J_7=-2\displaystyle\int\limits_{\mathbb{R}^n}z_{x_1x_1}\sum_{i=1}^n v_{x_i}\triangle v\,dx,\\ J_8=-4\displaystyle\int\limits_{\mathbb{R}^n}z_{x_1x_1}v_{x_1}\triangle v\,dx, & J_9=-2\displaystyle\int\limits_{\mathbb{R}^n}z_{x_1x_1x_1}v\triangle v\,dx,\end{cases} \tag{3.2.18}$$

$$\bar{J}_3=-\int\limits_{\mathbb{R}^n}\big(|\nabla v|^2+|\triangle v|^2\big)\sum_{i=1}^n v_{x_i}\,dx-4\int\limits_{\mathbb{R}^n}\triangle v\sum_{i=1}^n\nabla v\cdot\nabla v_{x_i}\,dx.$$

By the Cauchy inequality, the terms in (3.2.18) can be estimated as follows:

$$J_6+J_7\leqslant C\big|z_{x_1}(t)\big|_\infty\big|\triangle v(t)\big|_2^2+C\big|z_{x_1x_1}(t)\big|_\infty\big(\big|\nabla v(t)\big|_2^2+\big|\triangle v(t)\big|_2^2\big), \tag{3.2.19}$$

$$J_8\leqslant C\big|z_{x_1x_1}(t)\big|_\infty\big(\big|v_{x_1}(t)\big|_2^2+\big|\triangle v(t)\big|_2^2\big) \tag{3.2.20}$$

and

$$\begin{aligned} J_9&\leqslant\frac12\int_{\mathbb{R}^n}|\triangle v|^2\,dx+2\int_{\mathbb{R}^n}z_{x_1x_1x_1}^2v^2\,dx\\ &\leqslant\frac12\int_{\mathbb{R}^n}|\triangle v|^2\,dx+C\big(\|v_0\|_1^2+\|w_0\|_3^2+M_2^2+\delta_0^2\big)\big|z_{x_1x_1x_1}(t)\big|_\infty^2. \end{aligned}\tag{3.2.21}$$

Substituting (3.2.19)–(3.2.21) into (3.2.17), then integrating the resulting equation over $(0,t)$, and using Lemma 3.2, (3.2.9) and the estimates on J_4, J_5, $\bar J_3$, we arrive at

$$\begin{aligned} &\int_{\mathbb{R}^n}\big(|\nabla v|^2+\big|\nabla^2v\big|^2\big)\,dx+\int_0^t\int_{\mathbb{R}^n}\big\{\big|\nabla^2v\big|^2+z_{x_1}\big(|\nabla v|^2+\big|\nabla^2v\big|^2\big)\big\}\,dx\,d\tau\\ &\quad\leqslant C\big(\|v_0\|_2^2+\|w_0\|_3^2+M_2^2+\delta_0^2\big). \end{aligned}\tag{3.2.22}$$

Multiplying (3.2.17) by $(1+t)^{\beta+1}$, then integrating the resulting equation over $(0,t)$, we have

$$\begin{aligned} &(1+t)^{\beta+1}\int_{\mathbb{R}^n}\big(|\nabla v|^2+|\triangle v|^2\big)\,dx+\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\beta+1}\big\{2|\triangle v|^2+z_{x_1}\big(|\nabla v|^2+|\triangle v|^2\big)\big\}\,dx\,d\tau\\ &\quad=\|\nabla v_0\|_1^2+\sum_{i=4}^9\int_0^t(1+\tau)^{\beta+1}J_i\,d\tau+\bar J_4, \end{aligned}\tag{3.2.23}$$

where

$$\bar J_4=(\beta+1)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^\beta\big(|\nabla v|^2+|\triangle v|^2\big)\,dx\,d\tau+\int_0^t(1+\tau)^{\beta+1}\bar J_3\,d\tau.$$

By (3.2.3), Lemma 3.2, (3.2.13) and (3.2.14), the terms on the right-hand side of (3.2.23) can be estimated as follows:

$$\int_0^t(1+\tau)^{\beta+1}J_4\,d\tau\leqslant C\int_0^t(1+\tau)^{\beta+1}\big|z_{x_1}(\tau)\big|_\infty\big|\nabla v(\tau)\big|_2^2\,d\tau\leqslant C\big(1+(1+t)^{\beta-\frac n2}\big), \tag{3.2.24}$$

$$\begin{aligned} \int_0^t(1+\tau)^{\beta+1}J_5\,d\tau&\leqslant\int_0^t(1+\tau)^\beta\big|v_{x_1}(\tau)\big|_2^2\,d\tau+\int_0^t(1+\tau)^{\beta+2}\big|z_{x_1x_1}(\tau)\big|_\infty^2\big|v(\tau)\big|_2^2\,d\tau\\ &\leqslant C\big(1+(1+t)^{\beta-\frac n2}\big), \end{aligned}\tag{3.2.25}$$

$$\int_0^t (1+\tau)^{\beta+1} J_6 \, d\tau \leqslant C \int_0^t (1+\tau)^{\beta+1} \big|z_{x_1}(\tau)\big|_\infty \big|\triangle v(\tau)\big|_2^2 \, d\tau$$
$$\leqslant C\big(\|w_0\|_2 + M_1 + \delta_0^{\frac{1}{2}}\big)^{\frac{1}{2}} \int_0^t (1+\tau)^{\beta+1} \big|\triangle v(\tau)\big|_2^2 \, d\tau, \tag{3.2.26}$$

$$\int_0^t (1+\tau)^{\beta+1} (J_7 + J_8) \, d\tau$$
$$\leqslant \frac{1}{8} \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\beta+1} |\triangle v|^2 \, dx \, d\tau + C \int_0^t (1+\tau)^{\beta+1} \big|z_{x_1 x_1}(\tau)\big|_\infty^2 \big|\nabla v(\tau)\big|_2^2 \, d\tau$$
$$\leqslant \frac{1}{8} \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\beta+1} |\triangle v|^2 \, dx \, d\tau + C(1+t)^{\beta - \frac{n}{2}}, \tag{3.2.27}$$

$$\int_0^t (1+\tau)^{\beta+1} J_9 \, d\tau \leqslant \frac{1}{8} \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\beta+1} |\triangle v|^2 \, dx \, d\tau + C \int_0^t (1+\tau)^{\beta+1} \big|z_{x_1 x_1 x_1}(\tau)\big|_\infty^2 \big|v(\tau)\big|_2^2 \, d\tau$$
$$\leqslant \frac{1}{8} \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\beta+1} |\triangle v|^2 \, dx \, d\tau + C(1+t)^{\beta - \frac{n}{2}}. \tag{3.2.28}$$

Substituting (3.2.24)–(3.2.28) into (3.2.23) and using the estimate on $\bar{J}_4$, we have for $\beta > \frac{n}{2}$

$$(1+t)^{\beta+1} \int_{\mathbb{R}^n} \big(|\nabla v|^2 + \big|\nabla^2 v\big|^2\big) \, dx + \int_0^t \int_{\mathbb{R}^n} (1+\tau)^{\beta+1} \big\{\big|\nabla^2 v\big|^2 + z_{x_1}\big(|\nabla v|^2 + \big|\nabla^2 v\big|^2\big)\big\} \, dx \, d\tau$$
$$\leqslant C\big(1 + (1+t)^{\beta - \frac{n}{2}}\big). \tag{3.2.29}$$

Consequently, since $\beta > \frac{n}{2}$ in (3.2.29), we obtain

$$\big\|\nabla v(t)\big\|_1 \leqslant C(1+t)^{-(\frac{n}{4} + \frac{1}{2})}. \tag{3.2.30}$$

Step 3. Multiplying $\triangle(3.2.2)_1$ by $2\triangle v$, using (3.2.16), we have

$$\big(|\triangle v|^2 + \big|\nabla(\triangle v)\big|^2\big)_t + 2\big|\nabla(\triangle v)\big|^2 + |\triangle v|^2 \sum_{i=1}^n v_{x_i} + z_{x_1} |\triangle v|^2 + 4\triangle v \sum_{i=1}^n \nabla v \cdot \nabla v_{x_i}$$
$$+ 2 z_{x_1 x_1} \sum_{i=1}^n v_{x_i} \triangle v + 4 z_{x_1} \sum_{i=1}^n v_{x_1 x_i} \triangle v + 4 z_{x_1 x_1} v_{x_1} \triangle v + 2 z_{x_1 x_1 x_1} v \triangle v$$
$$- 2\triangle v \triangle^2 \left(v \sum_{i=1}^n v_{x_i} \right) - 2\triangle v \triangle^2 \left(\sum_{i=1}^n (zv)_{x_i} \right) - 2\,\mathrm{div}\big\{\triangle v \nabla(\triangle v + \triangle v_t)\big\}$$
$$+ \sum_{i=1}^n \big\{(z+v)|\triangle v|^2\big\}_{x_i} = 0. \tag{3.2.31}$$

Rewrite the eleventh term on the left-hand side of (3.2.31) as follows:

$$
\begin{aligned}
-2\triangle v\triangle^2\left(\sum_{i=1}^n z v_{x_i}+z_{x_1}v\right) &= -2\triangle v\triangle\left(\sum_{i=1}^n (z_{x_1x_1}v_{x_i}+2z_{x_1}v_{x_1x_i}+z\triangle v_{x_i})\right)\\
&\quad -2\operatorname{div}\big\{\triangle v\nabla(z_{x_1x_1x_1}v+2z_{x_1x_1}v_{x_1}+z_{x_1}\triangle v)\big\}\\
&\quad +2\nabla(\triangle v)\cdot\nabla(z_{x_1x_1x_1}v+2z_{x_1x_1}v_{x_1}+z_{x_1}\triangle v). \qquad (3.2.32)
\end{aligned}
$$

Substituting (3.2.32) into (3.2.31), then integrating the resulting equation over $\mathbb{R}^n$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\triangle v|^2+\left|\nabla(\triangle v)\right|^2\right)dx+\int_{\mathbb{R}^n}\left\{2\left|\nabla(\triangle v)\right|^2+z_{x_1}\left(|\triangle v|^2+\left|\nabla(\triangle v)\right|^2\right)\right\}dx\\
&\quad=\sum_{i=6}^{13}J_i+\bar{J}_5, \qquad (3.2.33)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{10}&=-4\int_{\mathbb{R}^n}z_{x_1}\sum_{i=1}^n\nabla(\triangle v)\cdot\nabla v_{x_1x_i}\,dx-2\int_{\mathbb{R}^n}z_{x_1}\triangle v_{x_1}\sum_{i=1}^n\triangle v_{x_i}\,dx,\\
J_{11}&=-2\int_{\mathbb{R}^n}z_{x_1x_1}\left(\sum_{i=1}^n\nabla(\triangle v)\cdot\nabla v_{x_i}+\triangle v_{x_1}\triangle v\right)dx\\
&\quad-4\int_{\mathbb{R}^n}z_{x_1x_1}\left(\triangle v_{x_1}\sum_{i=1}^n v_{x_1x_i}+\nabla v_{x_1}\cdot\nabla(\triangle v)\right)dx,\\
J_{12}&=-2\int_{\mathbb{R}^n}z_{x_1x_1x_1}\left(\nabla v\cdot\nabla(\triangle v)+\triangle v_{x_1}\sum_{i=1}^n v_{x_i}\right)dx-4\int_{\mathbb{R}^n}z_{x_1x_1x_1}v_{x_1}\triangle v_{x_1}\,dx,\\
J_{13}&=-2\int_{\mathbb{R}^n}\left(\partial_{x_1}^4 z\right)v\triangle v_{x_1}\,dx,\\
\bar{J}_5&=-4\int_{\mathbb{R}^n}\triangle v\sum_{i=1}^n\nabla v\cdot\nabla v_{x_i}\,dx-\int_{\mathbb{R}^n}|\triangle v|^2\sum_{i=1}^n v_{x_i}\,dx-\int_{\mathbb{R}^n}\left|\nabla(\triangle v)\right|^2\sum_{i=1}^n v_{x_i}\,dx\\
&\quad-2\int_{\mathbb{R}^n}\sum_{i,j=1}^n(v_{x_j}\triangle v_{x_j}+v_{x_ix_j}\triangle v)\triangle v_{x_i}\,dx-4\int_{\mathbb{R}^n}\sum_{i,j,k=1}^n(v_{x_ix_k}v_{x_jx_k}+v_{x_k}v_{x_ix_jx_k})\triangle v_{x_j}\,dx.
\end{aligned}
$$

One easily gets

$$J_{10}\leqslant C\left|z_{x_1}(t)\right|_\infty\left|\nabla(\triangle v)(t)\right|_2^2, \qquad (3.2.34)$$

$$J_{11}\leqslant C\left|z_{x_1x_1}(t)\right|_\infty\left(\left|\nabla(\triangle v)(t)\right|_2^2+\left|\triangle v(t)\right|_2^2\right), \qquad (3.2.35)$$

$$J_{12}\leqslant C\left|z_{x_1x_1x_1}(t)\right|_\infty\left(\left|\nabla(\triangle v)(t)\right|_2^2+\left|\nabla v(t)\right|_2^2\right). \qquad (3.2.36)$$

It remains to estimate J_{13}. Since

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\partial_{x_1}^4 z\right)^2 v^2\,dx &\leqslant \int_{\mathbb{R}^{n-1}} \left\{\max_{x_1\in\mathbb{R}} v^2\left(\cdot,x',t\right) \int_{\mathbb{R}} \left(\partial_{x_1}^4 z(x_1,t)\right)^2 dx_1\right\} dx' \\
&\leqslant \left|\partial_{x_1}^4 z(t)\right|_2^2 \int_{\mathbb{R}^{n-1}} \max_{x_1\in\mathbb{R}} v^2\left(\cdot,x',t\right) dx' \\
&\leqslant C\left|\partial_{x_1}^4 z(t)\right|_2^2 \int_{\mathbb{R}^{n-1}} \left\|v\left(\cdot,x',t\right)\right\|_{L^2(\mathbb{R}_{x_1})} \left\|v_{x_1}\left(\cdot,x',t\right)\right\|_{L^2(\mathbb{R}_{x_1})} dx' \\
&\leqslant C\left|\partial_{x_1}^4 z(t)\right|_2^2 \int_{\mathbb{R}^{n-1}} \left\{\left\|v\left(\cdot,x',t\right)\right\|_{L^2(\mathbb{R}_{x_1})}^2 + \left\|v_{x_1}\left(\cdot,x',t\right)\right\|_{L^2(\mathbb{R}_{x_1})}^2\right\} dx' \\
&\leqslant C\left|\partial_{x_1}^4 z(t)\right|_2^2 \int_{\mathbb{R}^n} \left(v^2+v_{x_1}^2\right) dx,
\end{aligned}$$

where $x=(x_1,x')$ and $dx=dx_1\,dx'$, thus

$$\begin{aligned}
J_{13} &\leqslant \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla(\triangle v)\right|^2 dx + C\int_{\mathbb{R}^n} \left(\partial_{x_1}^4 z\right)^2 v^2\,dx \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla(\triangle v)\right|^2 dx + C\left|\partial_{x_1}^4 z(t)\right|_2^2 \int_{\mathbb{R}^n} \left(v^2+v_{x_1}^2\right) dx. \qquad (3.2.37)
\end{aligned}$$

Substituting (3.2.34)–(3.2.37) into (3.2.33), then integrating the resulting equation over $(0,t)$ and using Lemma 3.2, (3.2.22) and the estimates on J_6–J_9, $\bar{J}_5$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\left|\nabla^2 v\right|^2+\left|\nabla^3 v\right|^2\right) dx + \int_0^t\int_{\mathbb{R}^n} \left\{\left|\nabla^3 v\right|^2 + z_{x_1}\left(\left|\nabla^2 v\right|^2+\left|\nabla^3 v\right|^2\right)\right\} dx\,d\tau \\
&\quad\leqslant C\left(\|v_0\|_3^2+\|w_0\|_4^2+M_3^2+\delta_0^2\right). \qquad (3.2.38)
\end{aligned}$$

Multiplying (3.2.33) by $(1+t)^{\beta+2}$, then integrating the resulting equation over $(0,t)$, we get

$$\begin{aligned}
&(1+t)^{\beta+2}\int_{\mathbb{R}^n} \left(|\triangle v|^2+\left|\nabla(\triangle v)\right|^2\right) dx + 2\int_0^t\int_{\mathbb{R}^n} (1+\tau)^{\beta+2}\left|\nabla(\triangle v)\right|^2 dx\,d\tau \\
&\quad+\int_0^t\int_{\mathbb{R}^n} (1+\tau)^{\beta+2} z_{x_1}\left(|\triangle v|^2+\left|\nabla(\triangle v)\right|^2\right) dx\,d\tau \\
&\quad=\|\triangle v_0\|_1^2+\sum_{i=6}^{13}\int_0^t (1+\tau)^{\beta+2} J_i\,d\tau+\bar{J}_6, \qquad (3.2.39)
\end{aligned}$$

where

$$\bar{J}_6 = (\beta+2)\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\beta+1}\big(|\triangle v|^2 + \big|\nabla(\triangle v)\big|^2\big)\,dx\,d\tau + \int_0^t (1+\tau)^{\beta+2}\bar{J}_5\,d\tau.$$

By Lemma 3.2, (3.2.13), (3.2.14), (3.2.29) and (3.2.30), we obtain

$$\int_0^t (1+\tau)^{\beta+2} J_6\,d\tau \leqslant C\int_0^t (1+\tau)^{\beta+2}\big|z_{x_1}(\tau)\big|_\infty\big|\triangle v(\tau)\big|_2^2\,d\tau \leqslant C\big(1+(1+t)^{\beta-\frac{n}{2}}\big), \tag{3.2.40}$$

$$\begin{aligned}\int_0^t (1+\tau)^{\beta+2}(J_7+J_8)\,d\tau &\leqslant \int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\beta+1}|\triangle v|^2\,dx\,d\tau \\ &\quad + C\int_0^t (1+\tau)^{\beta+3}\big|z_{x_1x_1}(\tau)\big|_\infty^2\big|\nabla v(\tau)\big|_2^2\,d\tau \\ &\leqslant C\big(1+(1+t)^{\beta-\frac{n}{2}}\big),\end{aligned} \tag{3.2.41}$$

$$\begin{aligned}\int_0^t (1+\tau)^{\beta+2} J_9\,d\tau &= 2\int_0^t\int_{\mathbb{R}^n}(1+\tau)^{\beta+2} z_{x_1x_1} v\triangle v_{x_1}\,dx\,d\tau - \frac{1}{2}\int_0^t (1+\tau)^{\beta+2} J_8\,d\tau \\ &\leqslant \frac{1}{8}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C\int_0^t (1+\tau)^{\beta+2}\big|z_{x_1x_1}(\tau)\big|_\infty^2\big|v(\tau)\big|_2^2\,d\tau \\ &\quad + C\big(1+(1+t)^{\beta-\frac{n}{2}}\big) \\ &\leqslant \frac{1}{8}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C\big(1+(1+t)^{\beta-\frac{n}{2}}\big).\end{aligned} \tag{3.2.42}$$

In addition,

$$\begin{aligned}\int_0^t (1+\tau)^{\beta+2} J_{10}\,d\tau &\leqslant C\big|z_{x_1}(\tau)\big|_\infty\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau \\ &\leqslant C\big(\|w_0\|_2 + M_1 + \delta_0^{\frac{1}{2}}\big)^{\frac{1}{2}}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau,\end{aligned} \tag{3.2.43}$$

$$\begin{aligned}&\int_0^t (1+\tau)^{\beta+2}(J_{11}+J_{12})\,d\tau \\ &\quad\leqslant \frac{1}{4}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C\int_0^t (1+\tau)^{\beta+2}\big|z_{x_1x_1}(\tau)\big|_\infty^2\big|\triangle v(\tau)\big|_2^2\,d\tau\end{aligned}$$

$$+ C\int_0^t (1+\tau)^{\beta+2}\big|z_{x_1x_1x_1}(\tau)\big|_\infty^2\big|\nabla v(\tau)\big|_2^2\,d\tau$$

$$\leqslant \frac{1}{4}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C\big(1+(1+t)^{\beta-\frac{n}{2}}\big), \tag{3.2.44}$$

$$\int_0^t (1+\tau)^{\beta+2}J_{13}\,d\tau \leqslant \frac{1}{8}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C\int_0^t (1+\tau)^{\beta+2}\big|\partial_{x_1}^4 z(\tau)\big|_2^2\big\|v(\tau)\big\|_1^2\,d\tau$$

$$\leqslant \frac{1}{8}\int_0^t (1+\tau)^{\beta+2}\big|\nabla(\triangle v)(\tau)\big|_2^2\,d\tau + C(1+t)^{\beta-\frac{n}{2}}. \tag{3.2.45}$$

Substituting (3.2.40)–(3.2.45) into (3.2.39), and using the estimate on $\bar{J}_6$, we have for $\beta > \frac{n}{2}$

$$(1+t)^{\beta+2}\big\|\nabla^2 v(t)\big\|_1^2 + \int_0^t (1+\tau)^{\beta+2}\Big\{\big|\big(\sqrt{z_{x_1}}\nabla^2 v\big)(\tau)\big|_2^2 + \big|\big(\sqrt{z_{x_1}}\nabla^3 v\big)(\tau)\big|_2^2 + \big|\nabla^3 v(\tau)\big|_2^2\Big\}\,d\tau$$
$$\leqslant C\big(1+(1+t)^{\beta-\frac{n}{2}}\big). \tag{3.2.46}$$

Consequently, since $\beta > \frac{n}{2}$ in (3.2.46), we obtain

$$\big\|\nabla^2 v(t)\big\|_1 \leqslant C(1+t)^{-(\frac{n}{4}+1)}. \tag{3.2.47}$$

Step 4. First, similar to (2.1.55), $v(x,t)$ satisfies for $n = 2,3$

$$\big|v(t)\big|_\infty \leqslant C(1+t)^{-\frac{n}{2}}. \tag{3.2.48}$$

In addition, similar to (1.10*a*) in [14], $z(x_1,t)$ satisfies for $t \to \infty$

$$\big|z(\cdot,t) - u^R(\cdot/t)\big|_\infty \leqslant Ct^{-\frac{1}{2}}. \tag{3.2.49}$$

Thus, from (3.2.1) we have for $t \to \infty$

$$u(x,t) - u^R(x_1/t) = \big\{u(x,t) - z(x_1,t)\big\} + \big\{z(x_1,t) - u^R(x_1/t)\big\}$$
$$= v(x,t) + \big\{z(x_1,t) - u^R(x_1/t)\big\}$$

and

$$\big|u(\cdot,t) - u^R(\cdot/t)\big|_\infty \leqslant C\big|v(t)\big|_\infty + \big|z(\cdot,t) - u^R(\cdot/t)\big|_\infty \leqslant Ct^{-\frac{1}{2}}.$$

Similarly, we can also prove the *a priori* assumptions (3.2.3) hold. This completes the proof of Theorem 1.5. □

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References

[1] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. 102 (1975) 159–182.
[2] M.D. Francesco, Initial value problem and relaxation limits of the Hamer model for radiating gases in several space variables, NoDEA Nonlinear Differential Equations Appl. 13 (2007) 531–562.
[3] M.D. Francesco, C. Lattanzio, Optimal L^1 rate of decay to diffusion waves for the Hamer model of radiating gases, Appl. Math. Lett. 19 (2006) 1046–1052.
[4] W.L. Gao, C.J. Zhu, Asymptotic decay toward the planar rarefaction waves for a model system of the radiating gas in two dimensions, Math. Models Methods Appl. Sci. 18 (2008) 511–541.
[5] W.L. Gao, L.Z. Ruan, C.J. Zhu, Decay rates to the planar rarefaction waves for a model system of the radiating gas in n-dimensions, J. Differential Equations 244 (2008) 2614–2640.
[6] K. Hamer, Nonlinear effects on the propagation of sound waves in a radiating gas, Quart. J. Mech. Appl. Math. 24 (1971) 155–168.
[7] Y. Hattori, K. Nishihara, A note on the stability of the rarefaction wave of the Burgers equation, Japan J. Indust. Appl. Math. 8 (1991) 85–86.
[8] T. Hosono, S. Kawashima, Decay property of regularity-loss type and application to some nonlinear hyperbolic–elliptic system, Math. Models Methods Appl. Sci. 11 (2006) 1839–1859.
[9] L. Hsiao, T.-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys. 143 (1992) 599–605.
[10] F.M. Huang, R.H. Pan, Convergence rate for compressible Euler equations with damping and vacuum, Arch. Ration. Mech. Anal. 166 (2003) 359–376.
[11] T. Iguchi, S. Kawashima, On space–time decay properties of solutions to hyperbolic–elliptic coupled systems, Hiroshima Math. J. 32 (2002) 229–308.
[12] K. Ito, Asymptotic decay toward the planar rarefaction waves for viscous conservation laws in several dimensions, Math. Models Methods Appl. Sci. 6 (1996) 315–338.
[13] S. Kawashima, S. Nishibata, Shock waves for a model system of a radiating gas, SIAM J. Math. Anal. 30 (1999) 95–117.
[14] S. Kawashima, Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, Kyushu J. Math. 58 (2004) 211–250.
[15] C. Lattanzio, P. Marcati, Global well-posedness and relaxation limits of a model for radiating gas, J. Differential Equations 190 (2003) 439–465.
[16] H. Liu, E. Tadmor, Critical thresholds in a conservation model for nonlinear conservation laws, SIAM J. Math. Anal. 33 (2001) 930–945.
[17] T. Luo, Asymptotic stability of planar rarefaction waves for relaxation approximation of conservation laws in several dimensions, J. Differential Equations 133 (1997) 255–279.
[18] A. Matsumura, K. Nishihara, Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas, Comm. Math. Phys. 144 (1992) 325–335.
[19] Y. Naito, T. Sato, Positive solutions for semilinear elliptic equations with singular forcing terms, J. Differential Equations 235 (2007) 439–483.
[20] M. Nishikawa, K. Nishihara, Asymptotics toward the planar rarefaction wave for viscous conservation law in two space dimensions, Trans. Amer. Math. Soc. 352 (2000) 1203–1215.
[21] S. Schochet, E. Tadmor, The regularized Chapman–Enskog expansion for scalar conservation laws, Arch. Ration. Mech. Anal. 119 (1992) 95–107.
[22] D. Serre, L^1-stability of nonlinear waves in scalar conservation laws, in: Handbook of Differential Equations, vol. 1, Evolutionary Equations, 2004, pp. 473–553.
[23] Y. Tanaka, Asymptotic behavior of solutions to the one-dimensional model system for a radiating gas, Master's thesis, Kyushu University, 1995 (in Japanese).
[24] W.G. Vincenti, C.H. Kruger, Introduction to Physical Gas Dynamics, Wiley, New York, 1965.
[25] Z.P. Xin, Asymptotic stability of rarefaction waves for 2×2 viscous hyperbolic conservation laws – the two modes case, J. Differential Equations 78 (1989) 191–219.
[26] T. Yang, H.J. Zhao, BV estimates on Lax–Friedrichs' scheme for a model of radiating gas, Appl. Anal. 83 (2004) 533–539.
[27] H.J. Zhao, Nonlinear stability of strong planar rarefaction waves for the relaxation approximation of conservation laws in several space dimensions, J. Differential Equations 163 (2000) 198–223.
[28] C.J. Zhu, Global resolvability for a viscoelastic model with relaxation, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995) 1277–1285.
[29] C.J. Zhu, Asymptotic behavior of solutions for p-system with relaxation, J. Differential Equations 180 (2002) 273–306.